0.5 | Sign Diagrams

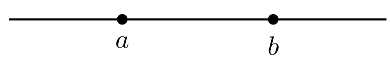
The sign diagram method covered in this worksheet is based on a custom method created by Dr. Erik Boczko for MATH1300 at Ohio University.

Sign diagrams are a method of analyzing functions that allows us to find on which intervals a given function is positive or negative. Traditionally, you may be used to drawing a number line, and then choosing test values and computing them to analyze outputs. This method does work, however for increasingly complicated functions this becomes increasingly time consuming, and sometimes basically requires the use of a calculator. The method shown in this paper allows you to compute the sign of an interval on a function without needing to plug in *any* values whatsoever. Let's construct a sign diagram for an arbitrary function, and then look at how we can interpret it.

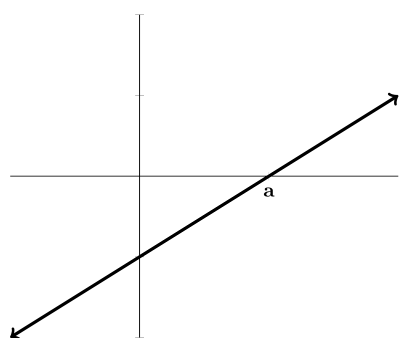
Constructing a Sign Diagram:

Let $f(x) = (x - a)(x - b)$

We begin by **identifying the zeros of the function**. Draw a number line and place points for the zeros in order. Our zeros here are a and b, and so for the sake of our example let's assume that $a < b$.

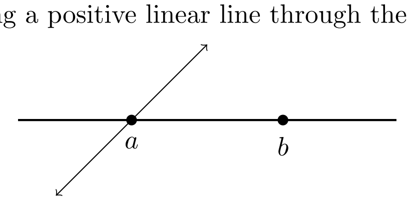


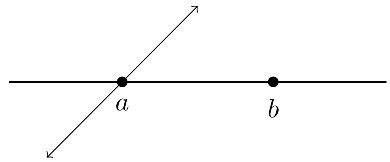
Now, we want to **look at the specific factor which generated each zero**. a was generated by the factor $(x - a)$. Let's imagine what that would look like should we graph it on a coordinate plane.



$y = x - a$ will be a positive linear function with an x-intercept at a. This means, that for any x-value to the left of a, the corresponding y value will be *negative*. Similarly, for any x-value to the right of a, the corresponding y value will be *positive*.

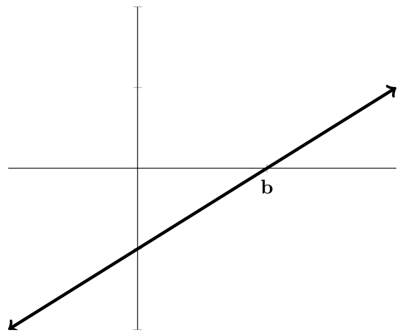
We will demonstrate this fact by drawing a positive linear line through the point a on our sign diagram.





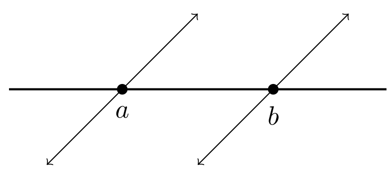
Making the line perfectly match the slope is irrelevant here, we are trying to convey the fact that all values to the left of a will be producing a **negative output**, and all values to the right of a will be producing a **positive output**.

We repeat this process again, asking what the factor that produced b looks like. b was generated by the factor $(x - b)$. Let's imagine this on the coordinate plane.



This function again, is positive and linear, and will have an x-intercept at b. All x-values to the left of b produce a negative output, and all x-values to the right of b will produce a positive output.

We represent this by drawing a positive linear line through b on our sign diagram.

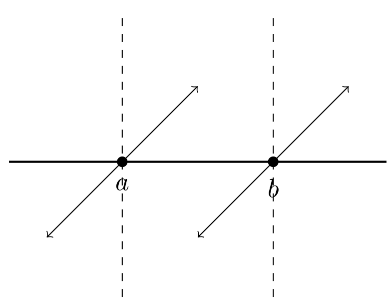


Once you have drawn lines for all points labeled on your diagram, the sign diagram is finished. We are now ready for the next step.

Interpreting a Sign Diagram:

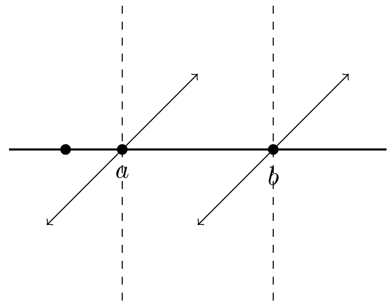
Keep in mind what a sign diagram is going to tell us. Our sign diagram is going to tell us on which intervals our function is positive, and which intervals our function is negative. This is also a good time to remind ourselves of the example function we are working with: $f(x) = (x - a)(x - b)$.

Notice that our diagram can be split into distinct intervals, separated by the points of interest we labeled.

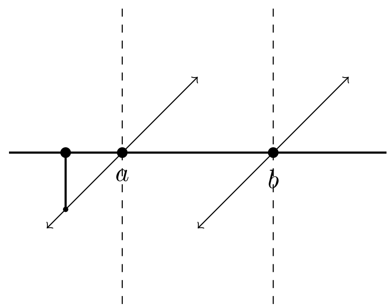


In many high-school courses, we would interpret this diagram by selecting sample points from each interval and then plugging this point back into our function. We will also be selecting sample points, however our points will be *arbitrary*, and we will need to make no calculations.

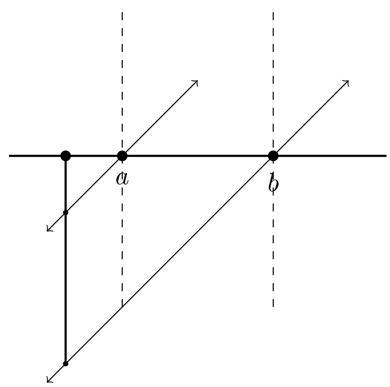
Pick an arbitrary point in the first interval ($< a$). You do *not* need to assign it any specific value. The one I have chosen is drawn as a dot on the number line.



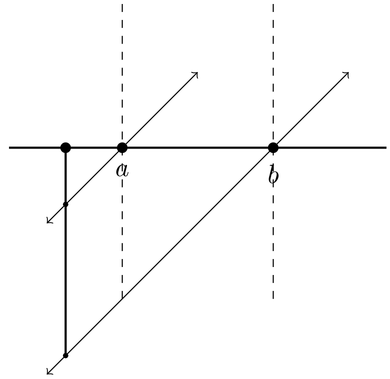
If we treat this arbitrary point as an input value, we want to find out **what is the sign of the output value provided by each factor within our function**. However we already did this work in the construction of the sign diagram. Any point to the left of a has a negative output, and any point to the left of b has a negative output. We can demonstrate this idea by drawing a vertical line from the point we labeled, and seeing which parts of the lines we drew intersect this vertical line.



As we start drawing this vertical line downward, we can see that the arbitrary point we chose will be generating a negative value from the factor $(x - a)$ (we know this because we are *below* the number line we drew). Let's keep drawing the line and see what other lines we intersect.



If we keep extending this vertical line downward from the arbitrary point, we see that we pick up another negative value from the factor $(x - b)$.

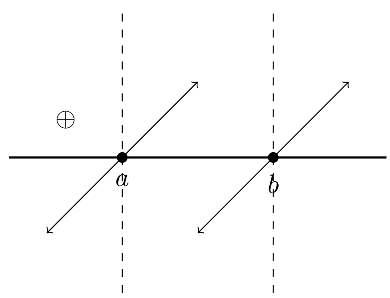


We are able to extend the line drawn from b due to the fact that lines will extend forever in both directions. So even if it seems like our vertical line might not touch the line drawn for b, we can extend the line as far as we need to make the intersection work.

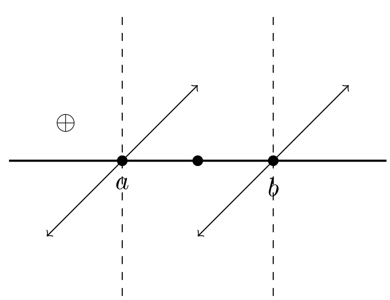
Once we have intersected all lines drawn, we will count up the number of positive and negative values generated by each. Remember: **if the intersection happens below the number line the value is negative, if the intersection happens above the number line the value is positive**.

We have two negative values generated by our lines. The function we are analyzing is $f(x) = (x - a)(x - b)$. Since our arbitrary values generated by each factor are negative, and the output of $f(x)$ can be represented by the multiplication of its factors, **we can calculate the sign of $f(x)$ on the interval we analyzed by multiplying the signs of the values generated**.

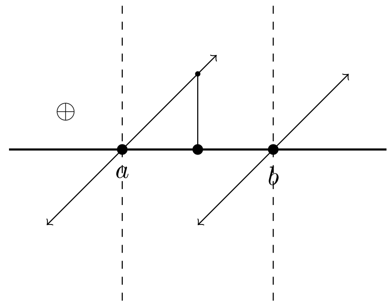
Notice that a negative value multiplied by a negative value is negative, and so the entire interval we analyzed is positive. We can now write a positive sign for the interval.



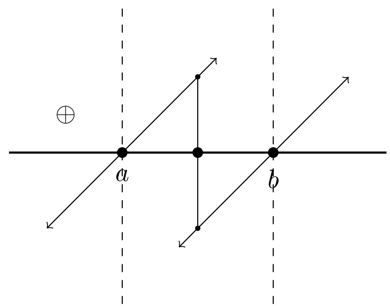
Now we repeat this process by choosing a new arbitrary point on the next interval.



We draw our vertical line once again, finding out which values will be captured by the intersections on the lines we drew earlier.

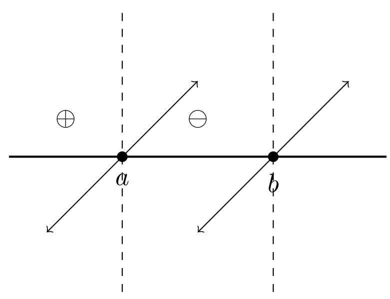


As we travel upwards we see that a *positive* value is generated by the line drawn through a. We know this value is *positive* because it is *above* the number line.

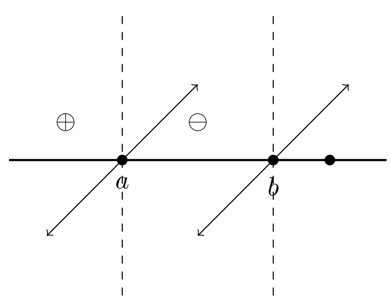


As we travel downwards we see that a *negative* value is generated by the line drawn through b. We know this value is *negative* because it is *below* the number line.

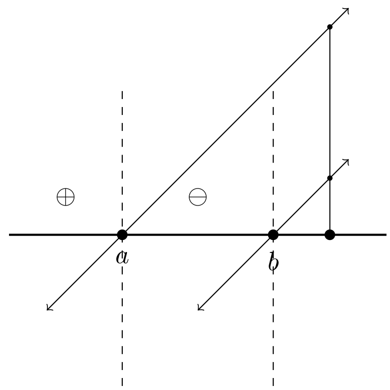
We have one positive value and one negative value generated by our lines on this interval. Recall that the output value of $f(x)$ can be represented by the multiplication of these values. We know that a positive times a negative is a negative, and we can label this interval as negative.



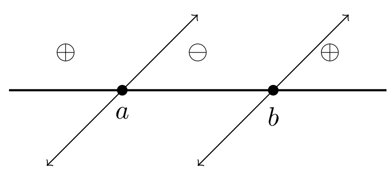
We can now choose an arbitrary point on our last interval to analyze.



As we extend a vertically upwards, we can see that we intersect both lines at points above the number line. This means that two positive values will be generated by these factors. Keep in mind we may extend the lines as needed to visualize this intersection.



We have two positive values generated by each factor, and we can represent the output of $f(x)$ over this interval as the multiplication of these factors. We know that a positive times a positive is a positive, so we can label this interval as positive.

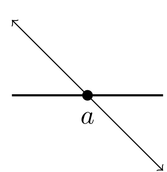


Now for the final interpretation: we have that $f(x) = (x - a)(x - b)$ is *positive* on the interval $(-\infty, a) \cup (b, \infty)$ and is *negative* on the interval (a, b).

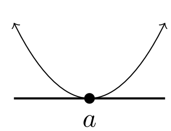
Drawing Lines for Different Factors:

In the example above, all of our factors were positive and linear. Let's look at a series of examples where other factors are used. Instead of entire unique functions being given, we will just look at how to draw these factors on the number line. The interpretation technique remains the same.

Negative Linear: If a factor looks like $(-x + a)$ we can draw



Parabolas: If a factor looks like $(x - a)^2$, we can simply draw a parabola on our point like so:

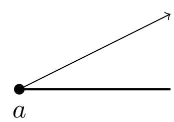


You might notice that both sides of the parabola are positive, and so they will not affect any sign values ($\oplus \cdot \oplus = \oplus$ and $\oplus \cdot \ominus = \ominus$). Therefore it is just as effective to leave the point of interest with *no lines drawn.* If the parabola was a negative factor however, we would want to draw a negative parabola as negative values will affect our interval.

Absolute Values: Like the case with parabolas, an absolute value will not produce a negative value that will affect our interval, therefore we may leave it blank (unless there is a negative sign specifically in front of the absolute value bars, then we can draw a negative absolute value).

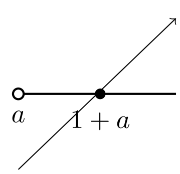
Higher Degree Factors: If a factor is of the form $(x - a)^n$. We can draw different lines depending on the value of n. If n is even, then we get something that looks like a parabola, and we can simply follow the guidelines for parabolas. If n is odd, then we actually can follow the normal procedure for lines. Keep in mind the specific output is not what is important in sign diagrams, it is the sign of the output which we care about. Therefore $(x - a)^3$ can simply be drawn as a straight line.

Root Functions: If a factor if of the form $\sqrt{x - a}$, then we can represent this as a positive line extending from the point of interest as root functions are always increasing. The one thing to be extra careful about however, it that root functions have restricted domains. $\sqrt{x - a}$ will have a domain of $[a, \infty)$. If one factor of a function has a non-existing domain, then the domain of the entire function must follow suit. Therefore if we include a root function as a factor it can be helpful to draw the point where the domain ends as an endpoint on our number line to demonstrate that no values exist past that point:



Exponential Functions: If a factor is of the form b^{x-a}, then we can leave the point of interest blank. Exponential functions are always positive, and so no change will happen to our signs. The only case in which we might want to represent with something, is if the factor is a negative exponential, then we could follow the procedure for negative parabolas to represent the idea that a negative is being applied everywhere.

Logarithmic Functions: If a factor is of the form $\log_b(x - a)$, then we will want to take note of two things. For one, the domain of the logarithm is restricted to (a, ∞), so just like the case of a root function, we may want to draw the point a as an endpoint to signify that the domain does not exist before that point. With logarithms, the end point is not included, so we can also signify this using an open circle. As for drawing the line, we want to note that the sign change in the logarithmic function actually happens at the point $1 + a$, and so we can separately draw this point of interest, and then represent a straight line though the point. If the logarithm was instead negative, we can handle that situation similar to how we have done in previous cases.



When to Use a Sign Diagram:

If you are reading through this paper, I would assume you already have an idea of where we use sign diagrams in a precalculus class. However, in case you are reading this for fun, let's briefly cover two cases where sign diagrams become very useful.

Inequalities: Sign diagrams are incredibly useful for solving inequalities. If you can set up an inequality into the form $f(x) < 0$ where $f(x)$ is fully factored, you can simply make and interpret a sign diagram to find where $f(x)$ is less than 0, in other words, where $f(x)$ is negative!

Graphing Rational Functions: If asked to graph a rational function of the from $\frac{f(x)}{g(x)}$, at first glance it can be very hard to determine which parts of the function are positive and which are negative due to the difficulty in determining end behavior in rational functions. However if $f(x)$ and $g(x)$ are both factorable we can simply construct a sign diagram and find intervals that are positive an negative for the whole function. Sign diagrams have almost no change in method for rational functions due to the fact that if $g(c)$ is negative then $\frac{1}{g(c)}$ will be negative as well.