2.1 | Graphs of Polynomial Functions

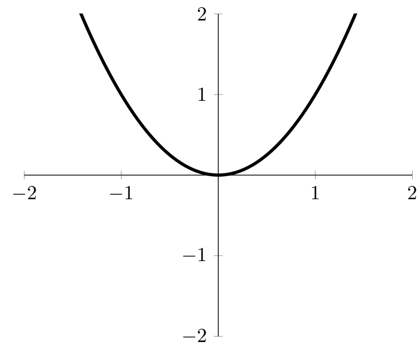
Monomial Functions: A monomial function is essentially a function with one "term." Nothing is separated out by + or − signs. Monomial functions come in the form:

$$f(x) = b \text{ or } f(x) = ax^n$$

where a and b are real numbers. The domain of a monomial is $(-\infty, \infty)$. There are many unique monomial functions, but we can place them in two general categories.

Even Functions: A function is said to be **even** if $f(-x) = f(x)$ for all x. In a monomial function, this is visually represented by a U shape on the graph. An example is shown to the right.

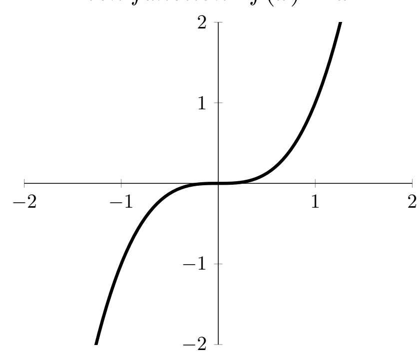
In the case of monomial functions, even functions will have the power x is raised to be an even number.



Even function: $f(x) = x^2$

Odd Functions: A function is said to be **odd** if $f(-x) = -f(x)$ for all x. In a monomial function, this is visually represented by an almost lightning bolt shape on the graph. An example is shown to the right.

In the case of monomial functions, odd functions will have the power x is raised to be an odd number.



Odd Function: $f(x) = x^3$

Transforming Functions: When we want to graph a monomial function, we can use a method known as function transformation, which looks at key values surrounding the monomial to move it around on the coordinate plane. The following theorem from the textbook goes into more detail.

Textbook Theorem 2.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = a(x-h)^n + k$ can be obtained from the graph of $f(x) = x^n$ by performing the following operations, in sequence.

1. add h to the x-coordinates of each of the point on the graph f. This results in a horizontal shit to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = x^n$ to $y = (x-h)^n$

2. multiply the y-coordinates of each of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = (x-h)^n$ to $y = a(x-h)^n$.

3. add k to the y-coordinates of each of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a(x-h)^n$ to $y = a(x-h)^n + k$.

The textbook theorem of course is very dense and can be hard to follow for those who aren't used to reading math textbooks. Before we look at a reasonable translation of the theorem however, we should discuss something called parent functions.

Parent Functions: The parent function to a given function is the underlying monomial function of the form $y = x^n$ which we can pick out if we ignore any extra values. For example: the function $g(x) = 2(x-3)^4 + 5$ has a parent function $p(x) = x^4$. We can then analyze how the original parent function was transformed on the graph by looking at the values surrounding it.

In English 2.1. Given a function of the form $F(x) = a(x-h)^n + k$, write down the parent function $P(x) = x^n$. Write down 3 *sample points* from the parent function, then do the following:

1. Add h to every x value.

2. Multiply all y values by a.

3. Add k to every y value.

Graph these new points, then trace the parent function.

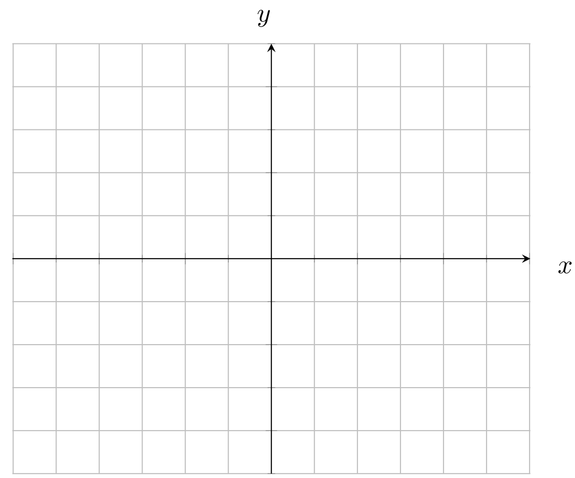
Picking Sample Points: When choosing sample points from a parent function (or any function in this course), we generally want to choose points that are *unique* enough to capture the overall shape of the function, but we also want to choose points that make our calculations as *simple* as possible. Below are recommended sample points for both even and odd monomials.

- **Even Functions**: Choose the sample points $\{(-1, 1), (0, 0), (1, 1)\}$

- **Odd Functions**: Choose the sample points $\{(-1, -1), (0, 0), (1, 1)\}$

A great advantage of monomials is that these sample points stay the same even for very large even/odd monomial functions!

1. **Worked Example:** Sketch a graph the function: $f(x) = 2(x+1)^3 - 1$.



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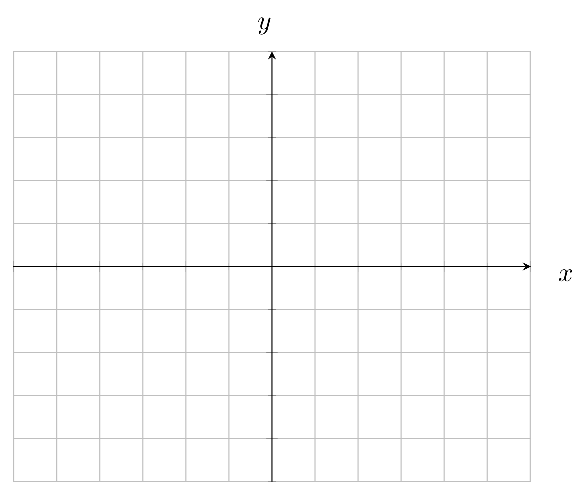
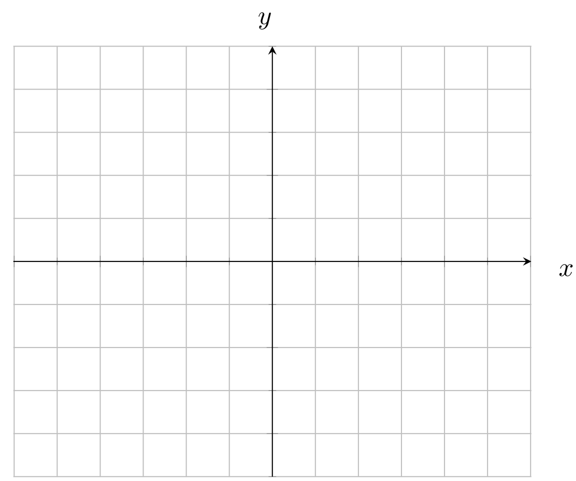
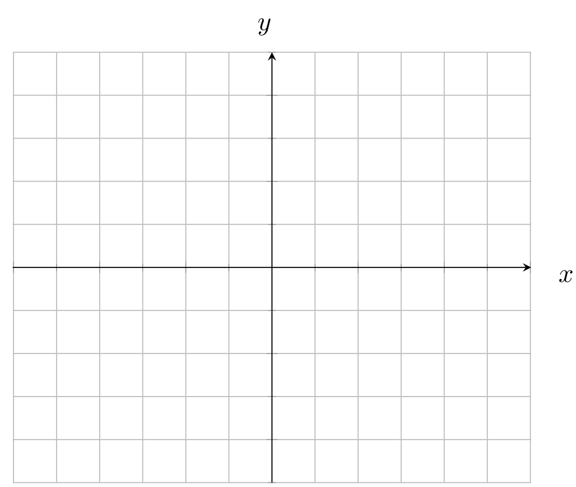
2. Sketch a graph of the function: $g(x) = 3(x-1)^2 + 1$



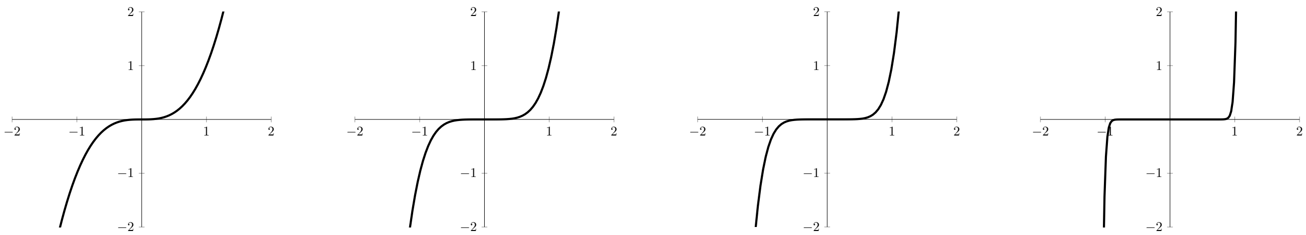
3. Sketch a graph of the function: $h(x) = \frac{1}{2}(x-1)^3$



4. Sketch a graph of the function: $f(x) = 3(x-2)^4 + 2$



Higher Degree Monomials: As monomials increase in degree, thier distinct size for even/odd functions doesn't change, but the curve gets much steeper. An example is given for odd functions, but the same process occurs for even functions.



$f(x) = x^3$ $\quad$ $f(x) = x^5$ $\quad$ $f(x) = x^7$ $\quad$ $f(x) = x^{33}$

End Behavior: The end behavior of a monomial functions describes the direction in which the function tends to as x grows infinitely large or infinitely small. We write end behavior in the following format:

$$\text{as } x \to ?, f(x) \to ?$$

Where we fill in "?" with values of $-\infty$ or ∞ depending on our function.

Textbook Theorem 2.2. End Behavior of Monomial Functions
Suppose $f(x) = ax^n$ where $a \neq 0$ is a real number and $n \in \mathbb{N}$.

- If n is even:
  - if $a > 0$, as $x \to -\infty, f(x) \to \infty$ and as $x \to \infty, f(x) \to \infty$
  - for $a < 0$, as $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to -\infty$
- If n is odd:
  - for $a > 0$, as $x \to -\infty, f(x) \to -\infty$ and as $x \to \infty, f(x) \to \infty$
  - for $a < 0$ as $x \to -\infty, f(x) \to \infty$ and as $x \to \infty, f(x) \to -\infty$

5. Write the end behavior of the function:
   $f(x) = x^8$

6. Write the end behavior of the function:
   $q(x) = -2x^5$

7. Write the end behavior of the function:
   $g(x) = 34x^{91}$

8. Write the end behavior of the function:
   $h(x) = \frac{1}{1000}x^{1000}$

Polynomial Functions: A polynomial function is of the form

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where all values of a are real numbers and n is a natural number. The domain of a polynomial is $(-\infty, \infty)$.

This might look really confusing to read (it is), but think of a polynomial as *a series of terms with a coefficient paired with to a variable that is raised to a power.* A more concrete example might look like:

$$x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6$$

Definitions: Lets look at an example polynomial and identify some key terms associated with it.

$$2x^3 - 3x^2 + 5x + 7$$

- **Degree**: The degree of a polynomial is the largest power that any variable is raised to. In the above example the degree is 3.
- **Leading Term**: The term which includes the variable of highest degree is the leading term. In the above example the leading term is $[2x^3]$.
- **Leading Coefficient**: The leading coefficient is the coefficient of the leading term. In the above example the leading coefficient is 2.
- **Constant Term**: The number with no variable is the constant term. In the above example the constant term is 7.

Textbook Theorem 2.3. End Behavior of Polynomial Functions:
The end behavior of a polynomial function $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ with $a_n \neq 0$ matches the end behavior of $y = a_nx^n$
That is the end behavior of a polynomial function is determined by its leading term.

In English 2.3. End Behavior of Polynomial Functions:
The end behavior of any polynomial matches the monomial term of the highest degree.

5. Write the end behavior of the polynomial
$3x^3 - x^2 + 4x - 1$

6. Write the end behavior of the polynomial
$-2x^4 - 7x^3 + x^2 + 8 - x + 8$

7. Write the end behavior of the polynomial
-$4x^2 - x + 2x^4$

8. Write the end behavior of the polynomial
$-x^7 + 146234x^6 + 7323x^5 + 527956x^4 + 821776x^3 + 56728x^2 + 38737x + 75728$

Factored Polynomials: Polynomials, like quadratics, can also be factored. This section will not cover factoring techniques, but all you need to know for now is that some polynomials can be written in a fully factored form, where it is listed as a series of terms multiplied together.

Zeros of Polynomials: If we have a fully factored polynomial function, we can find the zeros of the polynomial using the zero product property, similarly to how we used it to find zeros of quadratics in section 1.4. If a polynomial is of the form:

$$f(x) = (x-a)(x-b)(x-c)\dots(x-z)$$

Then we know that the zeros of the polynomial are $a, b, c, \dots, z$.

5. List the zeros of the following polynomial: $f(x) = (x-1)(x+1)(x-2)(x+2)$

6. List the zeros of the following polynomial: $g(x) = x(x+1)(x-3)$

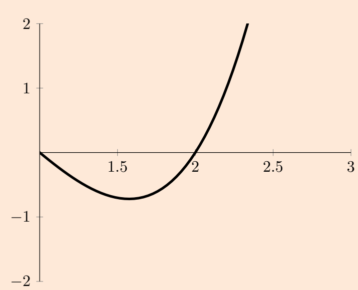
Multiplicity: Given a single term of a polynomial in the form $(x-c)^m$, the *multiplicity* of the term is given by the value of m. We can also say that the zero $x = c$ provided by the term has multiplicity m.

Textbook Theorem 2.5. The Role of Multiplicity: Suppose f is a polynomial function and $x = c$ is a zero of multiplicity m.

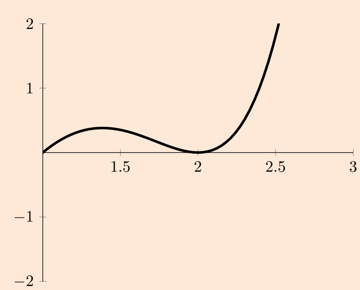
- If m is even, the graph of $y = f(x)$ touches and rebounds from the x-axis at $(c, 0)$.
- If m is odd, the graph of $y = f(x)$ crosses through the x-axis at $(c, 0)$.

In English 2.5. The Role of Multiplicity:
The multiplicity of a zero determines its behavior as it touches the x-axis.

- **Odd** multiplicity implies that the function strikes through the x-axis. An example at $x = 2$ is shown:



- **Even** multiplicity implies that the function touches and reflects off the x-axis. An example at $x = 2$ is shown:



7. List all zeros of the polynomial and their multiplicities: $f(x) = (x-3)(x+1)^3(x-4)^2$

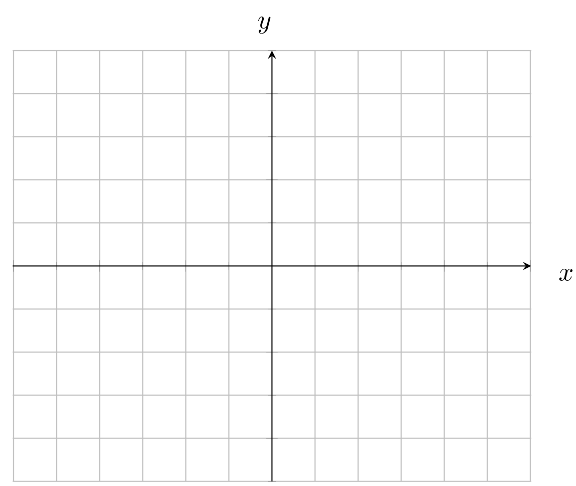
8. List all the zeros of the polynomial and their multiplicities: $g(x) = x^2(x-10)(x+12)^2$

Graphing Factored Polynomials: If we want to graph factored polynomials, we should find a few key points of information, and then use them to sketch a more complete graph.

- **Zeros**: Write down all zeros of the polynomial, plot the zeros on the x-axis
- **Multiplicity**: Look at each zero and determine what its behavior on the x-axis will look like using theorem 2.5.
- **End Behavior**: Use theorems 2.2 and 2.3 to mark down the end behavior of the polynomial. You may need to ask yourself what the polynomial would look like if we fully expanded it, as end behavior is not immediately obvious in a factored form.

We can then combine all these bits of information to sketch a fairly accurate[1] graph of a polynomial.

9. **Worked Example**: Sketch a graph of the polynomial $f(x) = (x+1)(x+2)^2(x+3)$



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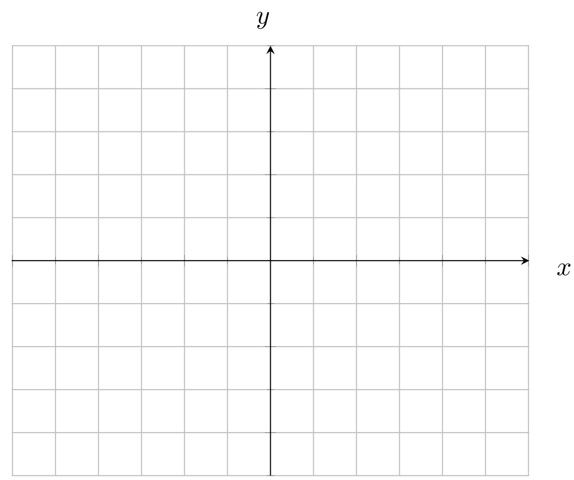
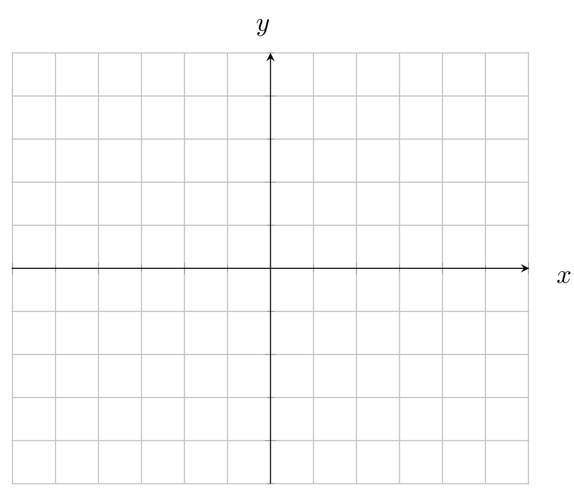
[1]Remember, you are not a computer. Your graphs don't need to be perfectly accurate. It is more important that you are able to identify zero and use multiplicity and end behavior to demonstrate your ability to imagine what the function might look like if graphed by a computer.

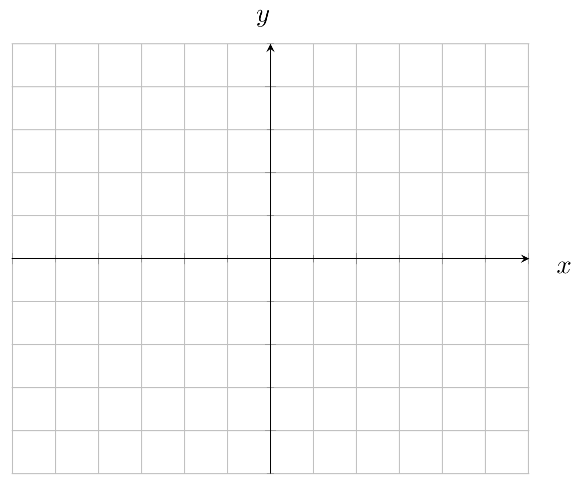
10. Sketch a graph of the polynomial: $a(x) = x(x+2)^2$



11. Sketch a graph of the polynomial: $g(t) = t(t+2)^3$



12. Sketch a graph of the polynomial: $P(z) = (z-1)(z-2)(z-3)(z-4)$



13. Sketch a graph of the polynomial: $h(t) = t^2(t-2)^2(t+2)^2$