2.2 | The Remainder and Factor Theorems

Long Division Warm Up: Before continuing with this worksheet, it is valuable to make sure that you know how to perform long division on two numbers, including the use of remainders.

1. Divide: $2024 \div 11$
2. Divide: $91301 \div 13$
3. Divide: $100000 \div 9$

You might notice that with any division problem, the divisor (the thing you divide by), the quotient (the answer), and the remainder (what is left over) are all related to one an other. Think about the following division problem: $86 \div 13$. In this case we get a solution of 6 with a reminder of 8. However, if we are just given the numbers $13, 6$, and 8, we can put them pack together to get 86 in the following manner: $86 = 13 \times 6 + 8$.

We can generalize this to any numbers we divide: $p \div d$. If $p \div d = q$ with remainder r, we can always write the expression for p as the following: $p = d \times q + r$. In this section, we expand this idea to polynomials.

Textbook Theorem 2.6. Polynomial Division:
Suppose $d(x)$ and $p(x)$ are nonzero polynomial functions where the degree of p is greater than or equal to the degree of d. There exists two unique polynomial functions, $q(x)$ and $r(x)$, such that $p(x) = d(x)q(x) + r(x)$, where either $r(x) = 0$ or the degree of r is strictly less than the degree of d.

In English 2.6. Polynomial Division:
In the same way that we can divide a larger number by a smaller one and write it in the form $p = dq + r$, we can divide a polynomial $p(x)$ by a polynomial of an equal or smaller degree and write the solution as: $p(x) = d(x)q(x) + r(x)$ where $q(x)$ is the quotient and $r(x)$ is the remainder.

In addition to Theorem 2.6, there are two supplemental theorems that we can use to our advantage for dividing polynomials, however don't worry too much about the details, they will become much clearer once we actually start dividing polynomials.

Textbook Theorem 2.7. The Remainder Theorem
Suppose p is a polynomial function of degree at least 1 and c is a real number. When $p(x)$ is divided by $x - c$ the remainder is $p(c)$. Said differently, there is a polynomial function $q(x)$ such that:

$$p(x) = (x - c)q(x) + p(c)$$

Textbook Theorem 2.8. The Factor Theorem
Suppose p is a nonzero polynomial function. The real number c is a zero of p if and only if $(x - c)$ is a factor of $p(x)$.

Now the big question is: how do we actually divide polynomials? To do this there are two main methods. We will cover both here. One thing to note, is that it is very hard to break down these methods into a list of steps that you can apply to any problem, and it is best to learn both through repeated examples.

Polynomial Long Division: The standard method of polynomial long division follows closley to that of standard long division[1]. Lets work through an example using the following polynomials:

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2}$$

First we want to write the polynomials in standard long division notation:

$$x - 2\overline{\big)\; x^3 + 4x^2 \;\; - 5x - 14}$$

Next, we ask ourselves: **what do we multiply x by to equal x^3?**. It is perfectly fine to ignore the -2 in the term $(x - 2)$ for now, we are simply concerend with finding a factor that we can multiply to the first term in our divisor to match the first term of the main polynomial. The answer here is of course, x^2, so we write it on top of the bar[2].

$$\begin{array}{r} x^2 \\ x - 2\overline{\big)\; x^3 + 4x^2 \;\; - 5x - 14} \end{array}$$

The next step is to actually multiply x^2 by the entire term $(x - 2)$. This yields: $x^2(x - 2) = x^3 - 2x^2$. Now with this new term, $x^3 - 2x^2$, we want to subtract it from the first two terms of the main polynomial. However, since we are subtracting the *polynomial* $x^3 - 2x^2$, we can represent the subtraction by multiplying out a (-1) and *adding* it to the first two terms of the main polynomial while paying attention to negatives where they occur. $(-1)(x^3 - 2x^2) = -x^3 + 2x^2$. Now we write this new equation underneath the main polynomial.

$$\begin{array}{r} x^2 \\ x - 2\overline{\big)\; x^3 + 4x^2 \;\; - 5x - 14} \\ \underline{- x^3 + 2x^2} \end{array}$$

[1]If you do not have a strong grasp on standard long division of numbers, please see the following YouTube video by *The Organic Chemistry Tutor*: `https://youtu.be/GiiuZ8sfw0O?si=B3y-1LFTSMk3LjEB`.

[2]The alignment of x^2 over top of the corresponding term $4x^2$ is not required, and is only done this way due to the software I am using to generate the long division equations. You can place the x^2 over top of the x^3 term if it makes things more organized for you.

Now simply add together the first two columns and bring down the next term. So $x^3+(-x^3)=0$ and $4x^2+2x^2=6x^2$, and finally we bring down the next term $-5x$.

$$\begin{array}{r|l} & x^2 \\ \hline x-2) & x^3+4x^2-5x-14 \\ & -x^3+2x^2 \\ \hline & 6x^2-5x \end{array}$$

Problem Solving Tip 1. You will know you have done everything correctly so far if the first terms (x^3 and $-x^3$ in our case) perfectly cancel to zero. If something goes wrong, in that an (x^3) term is still lingering around, then you almost certainly made a mistake.

Now, we simply repeat the process over again but we work based off the new polynomial we have written once bringing down the next term. We ask ourselves, what do we multiply x by to equal $6x^2$? The answer here is $6x$. This term is positive, so we write $+6x^2$ next to the x^2 from earlier.

$$\begin{array}{r|l} & x^2+6x \\ \hline x-2) & x^3+4x^2-5x-14 \\ & -x^3+2x^2 \\ \hline & 6x^2-5x \end{array}$$

Again, we multiply out $(x-2)$ by the term we just wrote down, so $(6x^2)(x-2)=6x^2-12x$. We will want to subtract this term however, so we multiply it by (-1) then write it below the new polynomial we are working from: $(-1)(6x^2-12x)=-6x+12x$.

$$\begin{array}{r|l} & x^2+6x \\ \hline x-2) & x^3+4x^2-5x-14 \\ & -x^3+2x^2 \\ \hline & 6x^2-5x \\ & -6x^2+12x \\ \hline \end{array}$$

Add the two polynomials together: $6x^2+(-6x^2)=0$, and $-5x+12x=7x$. Then bring down the next term -14.

$$\begin{array}{r|l} & x^2+6x \\ \hline x-2) & x^3+4x^2-5x-14 \\ & -x^3+2x^2 \\ \hline & 6x^2-5x \\ & -6x^2+12x \\ \hline & 7x-14 \end{array}$$

Repeat again. What multiplies by x to get $7x$?. We choose 7 here a write it next to the previous two terms on top of the bar.

$$
\begin{array}{r|rrrr}
 & x^2 & +6x & +7 & \\
\hline
x-2 & x^3+4x^2 & -5x & -14 & \\
 & -x^3+2x^2 & & & \\
\hline
 & 6x^2 & -5x & & \\
 & -6x^2+12x & & & \\
\hline
 & & 7x & -14 &
\end{array}
$$

It might seem like we have a complete answer, but we should still check the process to see if we get a remainder. We multiply 7 by $(x-2)$ to obtain $7x-14$. Then we multiply this by a (-1) to find: $(-1)(7x-14) = -7x+14$.

$$
\begin{array}{r|rrrr}
 & x^2 & +6x & +7 & \\
\hline
x-2 & x^3+4x^2 & -5x & -14 & \\
 & -x^3+2x^2 & & & \\
\hline
 & 6x^2 & -5x & & \\
 & -6x^2+12x & & & \\
\hline
 & & 7x & -14 & \\
 & & -7x & +14 & \\
\hline
\end{array}
$$

We find that when we add the last two lines together, we get 0. This means that we have no remainder.

$$
\begin{array}{r|rrrr}
 & x^2 & +6x & +7 & \\
\hline
x-2 & x^3+4x^2 & -5x & -14 & \\
 & -x^3+2x^2 & & & \\
\hline
 & 6x^2 & -5x & & \\
 & -6x^2+12x & & & \\
\hline
 & & 7x & -14 & \\
 & & -7x & +14 & \\
\hline
 & & & 0 &
\end{array}
$$

Now that we are finished, we can write the polynomial in the form $p(x) = d(x)q(x) + r(x)$. We have no remainder $(r(x) = 0)$ in this case, so we can simply write $p(x) = d(x)q(x)$:

$$x^3 + 4x^2 - 5x - 14 = (x-2)(x^2 + 6x + 7)$$

If you would like to check your work, simply expand the expression $d(x)q(x) + r(x)$ to verify that it equals $(p(x))$.

4. **Worked Example**: Use polynomial long division to divide: $(3x^2 - 2x + 1) \div (x - 1)$

Scan the QR code for a video solution.

5. Use polynomial long division to divide: $(4x^2 - 5x + 3) \div (x + 3)$

> **Problem Solving Tip 2.** Writing $0x^n$ in place of a missing term can help you to stay organized when dividing.

6. Use polynomial long division to divide: $(4x^3 + 2x - 3) \div (x - 3)$

Synthetic Division: The second main method to dividing polynomials is called synthetic division. The biggest advantage of synthetic division, is that it is much easier and faster to compute than that of standard polynomial long division. The biggest downside however, is that *synthetic division only works when dividing by a linear term.* This means that if you want to divide by a polynomial larger than degree 1 (like a quadratic for example), then you will be forced to use standard long division.

Synthetic division also is best learned though example, and is difficult to write down into steps. Let's go through an example using the exact same polynomials as before:

$$\frac{x^3 + 4x^2 - 5x - 14}{x - 2}$$

The setup division looks like a large **L**, with the *zero of the term you are dividing* to the left[3]. In our case $x - 2 = 0$ implies that $x = 2$, and so we write 2 to the left of the **L** shape. Next we write down the *coefficients of the polynomial we want to divide by* and write them at the top of the L shape. Our polynomial is $x^3 + 4x^2 - 5x - 14$, and so the coefficients we write are $1, 4, -5$, and -14. If we are missing a value, in other words a coefficient is 0, we should also write that.

$$\begin{array}{r|rrrr} & 1 & 4 & -5 & -14 \\ 2 & & & & \\ \hline & & & & \end{array}$$

The first step is not very exciting, we simply bring down the first coefficient below the bottom bar of the **L** shape.

$$\begin{array}{r|rrrr} & 1 & 4 & -5 & -14 \\ 2 & & & & \\ \hline & 1 & & & \end{array}$$

The next step, is to take this value we brought down, and multiply it by the zero we wrote down to the left. In our case, we brought down a 1, and the zero is 2. We take $1 \times 2 = 2$. We take this new value, and write it below the next coefficient in line.

$$\begin{array}{r|rrrr} & 1 & 4 & -5 & -14 \\ 2 & & 2 & & \\ \hline & 1 & & & \end{array}$$

Now, we add together the column that contains the new value. We have a 4 stacked on top of a 2, so we can add $4 + 2 = 6$ and write the value of 6 below the bar.

$$\begin{array}{r|rrrr} & 1 & 4 & -5 & -14 \\ 2 & & 2 & & \\ \hline & 1 & 6 & & \end{array}$$

[3] Traditionally the zero is placed to the upper left, but the program I am using to generate these images places it lower.

Now we repeat the process. We multiply the zero (2), by the new value (6) to find $2 \times 6 = 12$ and then write this value underneath the next coefficient.

	1	4	−5	−14
2		2	12	
	1	6		

We then add together the column, $-5 + 12 = 7$, and write this value below the bar.

	1	4	−5	−14
2		2	12	
	1	6	7	

We repeat one more time. We take $2 \times 7 = 14$, and write this value in the last spot.

	1	4	−5	−14
2		2	12	14
	1	6	7	

Now we can add the last column together, getting $-14 + 14 = 0$ and writing it below the bar.

	1	4	−5	−14
2		2	12	14
	1	6	7	0

Now we have finished synthetic divison, but we need a way to interpret these values we have written down. The values we wrote below the bar are the important ones, $1, 6, 7$, and 0. **The last value we write down is always the remainder.** In our case the remainder is 0, just like we found using long division.

As for the remaining three values, we want to fill in the information for the type of polynomial we would expect to get from our division. We divided a degree 3 polynomial by a degree 1 polynomial. This means that we should end up with a degree 2 polynomial. We will then go through and write in variables for the three coefficients that are not the remainder $(1, 6, 7)$. We start with 1, which becomes x^2, then we move to 6 which becomes x, and finally 7 which stays as a positive 7. We take this new polynomial and write it out as one: $x^2 + 6x + 7$. If you go back and look at the solution to the long division example, you will find that we have arrived at the same answer.

Now that we are finished, we can write the polynomial in the form $p(x) = d(x)q(x) + r(x)$. We have no remainder $(r(x) = 0)$ in this case, so we can simply write $p(x) = d(x)q(x)$:

$$x^3 + 4x^2 - 5x - 14 = (x - 2)(x^2 + 6x + 7)$$

If you would like to check your work, simply expand the expression $d(x)q(x) + r(x)$ to verify that it equals $(p(x))$.

Problem Solving Tip 3. When dividing a degree n polynomial by a degree m polynomial the resulting polynomial will have degree $n - m$.

7. **Worked Example**: Use synthetic division to divide: $(3x^2 - 2x + 1) \div (x - 1)$

Scan the QR code for a video solution.

Note: For the rest of the practice problems, you may choose which method you use: polynomial long division or synthetic division. Whichever method you find to be easiest you should default to. However, keep in mind that **synthetic division will not work when dividing by a polynomial greater than degree 1**, and so you should be proficient in standard long division for problems that require it.

8. Divide the polynomials: $(-2x^2 - 4x + 3) \div (x + 1)$

9. Divide the polynomials: $(x^2 - 5) \div (x - 5)$

10. Divide the polynomials: $(x^3 + 2x^2 - 3x + 4) \div (x - 7)$

11. Divide the polynomials: $(z^3 + 8) \div (z + 2)$

12. Divide the polynomials: $(x^3 + x^2 + x + 1) \div (x + 9)$

13. Divide the polynomials: $(7x^3 - 1) \div (x + 2)$

14. Divide the polynomials: $(5x^4 + x^2 - 8x + 2) \div (x - 4)$

15. Divide the polynomials: $(2x^5 + x^4 - 6x + 9) \div (x^2 - 3x + 1)$

Challenge Problem: A polynomial $p(x) = x^3 - 6x^2 + 11x - 6$ has the real zero $c = 1$. Find the remaining real zeros of $p(x)$.

Scan the QR for a video solution

Materials in SI are not a suitable replacement for materials in class. These materials are not for use on exams.