2.3 | Real Zeros of Polynomials

The big goal of chapter 2 is to be able to factor and graph polynomial functions. We already have a lot of important skills that will let us do this. In section 2.1 we learned how to graph polynomials that are already in factored form. In 2.2 we learned how to divide polynomials by factors and reduce them to a smaller degree. The only thing we are missing from being able to completely factor a polynomial, is *a method by which to identify possible roots* of a polynomial. In this section we will scope out a variety of methods that can be used to identify roots of a polynomial function.

Textbook Theorem 2.10. Cauchy's Bound: Suppose $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$. Let M be the largest of the numbers: $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \ldots, \frac{|a_{n-1}|}{|a_n|}$. Then all real zeros of f lie in the interval $[-(M+1), M+1]$.

In English 2.10. Cauchy's Bound: Given a polynomial, identify the *leading* coefficient and the *largest*[a] coefficient.[b] Calculate $M = \dfrac{|\text{largest coefficient}|}{|\text{leading coefficient}|}$. Then all zeros lie in the interval $[-(M+1), (M+1)]$.

[a]Take the largest in *magnitude*, do not take negatives or positives into consideration.
[b]If the largest coefficient happens to be the leading coefficient, take the second largest instead.

1. Use Cauchy's bound to identify an interval in which all zeros of the following polynomial lie: $2x^4+4x^3-x^2-6x-3$

2. Use Cauchy's bound to identify an interval in which all zeros of the following polynomial lie: $3x^4-x^3-4x^2+x-5$

Knowing where zeros lie is nice, but what we really want is a list of zeros that we can work with. This next theorem is the most important of the entire chapter, it will be very valuable to get good at using it.

Textbook Theorem 2.11. Rational Zeros Theorem: Suppose $f(x) = a_n x^n + a_{n_1} x^{n-1} + \cdots + a_1 x + a_0$ is a polynomial of degree n with $n \geq 1$, and $a_0, a_1, \ldots, a_n$ are integers. If r is a rational zero of f, then r is of the form $\pm\frac{p}{q}$ where p is a factor of the constant term a_0 and q is a factor of the leading coefficient a_n.

In English 2.11. Rational Zeros Theorem: Given a polynomial, write the factors of the leading coefficient, these values will become candidates for q. Write the factors of the constant term, these values will become candidates for p. Write out a list of all possible combinations of $\pm\frac{p}{q}$. This list contains potential rational roots for the polynomial.

3. Write the list of possible rational roots for the polynomial: $2x^4 + 4x^3 - x^2 - 6x - 3$

4. Write the list of possible rational roots for the polynomial: $2x^3 - 3x + 5$

One important thing to note about the rational zeros theorem is that not every value in our list $\pm\frac{p}{q}$ will actually be a root.[1] We instead need to *test* these values. The quickest method of doing so is to set up a synthetic division table and divide the root by the polynomial. If the remainder is zero, you will know you have found a root. Any other value of a remainder means the value is *not* a root.

As you might be able to tell, larger values for the leading coefficient and constant term can make the list $\pm\frac{p}{q}$ grow quite quickly. This is where the next two theorems of this section come into play, as they can help reduce the number of synthetic division calculations that you need to do overall.

[1] We can take this even further. It is possible that the rational roots theorem won't list *any of the function's roots at all.* In fact, the polynomial $2x^3 - 3x + 5$ is an example of this. Its only real zero is $x \approx -1.7188536202820$, which is not of the form $\pm\frac{p}{q}$ for this polynomial. However on an exam, you will never be given a polynomial which has no identifiable roots like this, unless it was to be used as an example similarly to how I just did.

Textbook Theorem 2.13. Descartes' Rule of Signs: Suppose $f(x)$ is the formula for a polynomial function written with descending powers of x.

- If P denotes the number of variations of sign in the formula for $f(x)$, then the number of positive real zeros (counting multiplicity) is one of the numbers $\{P, P-2, P-4, \ldots\}$
- If N denotes the number of variations of sign in the formula for $f(-x)$, then the number of negative real zeros (counting multiplicity) is one of the numbers $\{N, N-2, N-4, \ldots\}$

In English 2.13. Descartes' Rule of Signs: Write the terms of a polynomial $f(x)$ in the standard order, with the leading term first, and then constant term last.

- Reading the coefficients from left to right, count the number of times the sign changes. If the sign changes p times, then the number of positive real zeros for $f(x)$ is one from the following set: $\{p, p-2, p-4, \ldots\}$ with each value in this set being greater or equal to 0.
- Plug in $-x$ into $f(x)$, write out the new equation $f(-x)$. Read the new coefficients from left to right, and count the number of times the sign changes. If the sign changes n times, then the number of negative real zeros of $f(x)$ is one from the following set: $\{n, n-2, n-4, \ldots\}$ with each value in this set being greater or equal to 0.

The Descartes' Rule of Signs is not the most powerful theorem by any means. It is not necessary to use in every instance of searching for rational roots. However, it can sometimes be useful. For example, say that the theorem tells you that you have either 0 or 2 positive real roots. If you find one positive root on your own, then you know there is at least one other you have not found yet. The theorem is not much more useful outside of cases like these.

Textbook Theorem 2.14. Upper and Lower Bounds: Suppose f is a polynomial of degree $n \geq 1$.

- If $c > 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau have the same signs, then c is an upper bound for the real zeros of f. That is, there are no real zeros greater than c.
- If $c < 0$ is synthetically divided into f and all of the numbers in the final line of the division tableau alternate signs, then c is a lower bound for the real zeros of f. That is, there are no real zeros less than c.

NOTE: If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as $(+)$ or $(-)$ as needed.

The theorem for upper and lower bounds is not *required* for factoring polynomials, but is certainly useful. When working with the rational zeros theorem you will already be doing many instances of synthetic division, and so it is never a bad idea to keep this theorem in the back of your mind in case an instance of it shows up. Sometimes, for larger sets of $\pm\frac{p}{q}$, this theorem can quickly rule out lots of values and save you time on a problem.

As mentioned before, not all of the theorems listed are as important as the other. The general method of factoring polynomials goes as follows:

- Use the rational zeros theorem to find a list of test values.
- Use synthetic division (or another method you like) to test if a value is a root.
- Keep checking values until you have found enough roots to factor the polynomial.

The more you work on this process the more you will pickup on shortcuts along the way, but at the end of the day getting good at this takes *lots of practice*.

5. **Worked Example:** Factor the polynomial: $f(x) = x^3 - 2x^2 - 5x + 6$

Scan the QR code for a video solution

> **Problem Solving Tip 1.** When picking values from the rational zeros theorem to test, it can be beneficial to start with the ones that are closest to zero (or equal to zero) first. Not only will these values be the easiest to work with computationally, but you can also identify if one is an upper or lower bound using theorem **2.14**. That way you can save time by not testing any roots larger (for upper bounds) or smaller (for lower bounds).

6. Factor the polynomial: $p(x) = x^4 - 9x^2 - 4x + 12$

Problem Solving Tip 2. It can be useful to write out the resulting polynomial after finding a root using synthetic division. This new polynomial will share all roots with the original (except the one you just divided), and may potentially offer an easier scenario to use the rational zeros theorem on. The biggest advantage however, is if the resulting polynomial is a quadratic. At this point, you can simply try and factor the quadratic directly, those factors will then be zeros of the original polynomial.

7. Factor the polynomial: $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$

8. Factor the polynomial: $p(z) = z^3 + 4z^2 - 11z + 6$

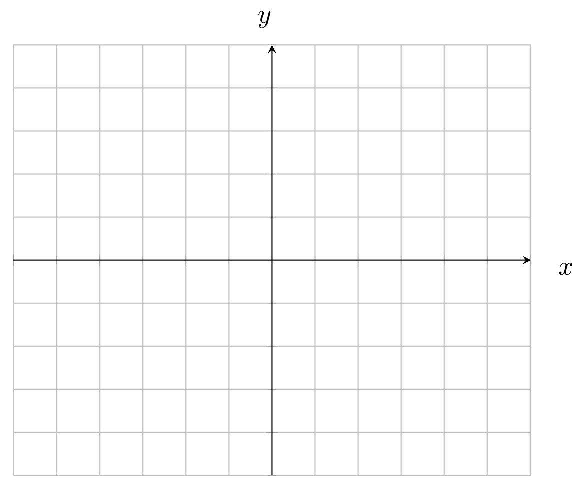
Problem Solving Tip 3. Sometimes it can be useful to do synthetic division twice with the same zero. After you complete the first division, do it again using the coefficients of the resulting polynomial. If a remainder of zero is once again found, then this zero has a multiplicity equal to the number of times you can successfully divide.

9. Factor the polynomial: $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

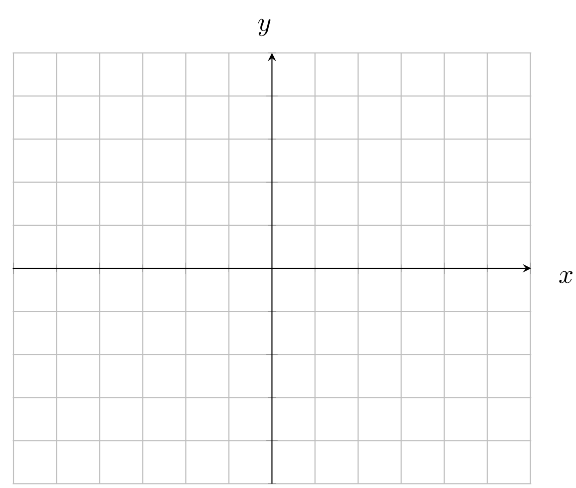
Problem Solving Tip 4. Sometimes, not every root of the polynomial will be captured by the list $\pm\frac{p}{q}$. In this case, try to reduce the polynomial down to a quadratic, and if the resulting quadratic cannot be factored, you can use the quadratic formula to solve for the remaining roots.

10. List all the zeros of the polynomial: $p(z) = 3z^3 + 3z^2 - 11z - 10$

11. Graph the polynomial: $x^4 - 5x^3 + 5x^2 + 5x - 6$



12. Graph the polynomial: $x^5 - 2x^4 - 2x^3 + 4x^2 + x - 2$



Solving Inequalities Involving Polynomials: In section 1.4 we looked at methods that can be used to solve inequalities involving quadratic functions. Those problems required that you factor a quadratic and then construct a sign diagram. In this section, we can use an almost identical process due to our ability to factor polynomial functions. To solve an inequality involving polynomial functions we can do the following:

- Move all terms to one side of the equation
- Factor the polynomial
- Make a sign diagram[2]
- Analyze the sign diagram and interpret based on the sign

13. **Worked Example:** Solve the inequality, express your answer in interval notation: $x^3 - 2x^2 - 5x + 6 \leq 0$

Scan the QR code for a video solution

14. Solve the inequality, express your answer in interval notation: $x^4 - 9x^2 \leq 4x - 12$

[2] If you are unfamiliar with how sign diagrams work, I have a dedicated worksheet for them on my website at `romansimkins.com/pal/worksheets`. I will use a sign diagram in the first worked example, so you are welcome to follow along there as well

15. Solve the inequality, express your answer in interval notation: $4z^3 \geq 3z + 1$

16. Solve the inequality, express your answer in interval notation: $3t^2 + 2t < t^4$

17. Solve the inequality, express your answer in interval notation: $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$