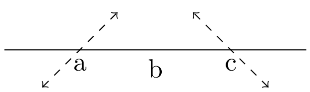
Signs Diagrams the Easy Way

Sign diagrams are important in precalculus for solving inequalities. They become important in calculus when implementing the first derivative test and the second derivative test for concavity. I have noticed that many students struggle with building sign diagrams and that they also rely heavily on evaluating the functions numerically. This becomes a real problem when the functions are transcendental. But if you learn the simple method described below you will never struggle again.

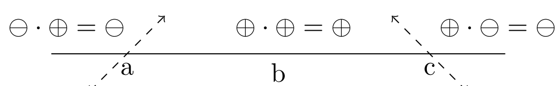
Here we show how to with polynomials. A sign diagram shows the SIGN of the OUTPUT-values of a function for each input value in the domain. Let's start with a polynomial that's already factored.

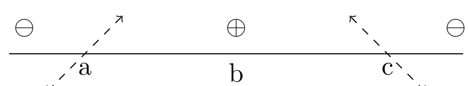
$$f(x) = (x - a)(x - b)^2(c - x)$$



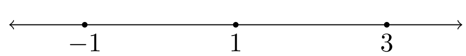
The term (x-a) is the **line** $y = x - a$, whose slope is 1 and x-intercept is $x = a$. The line is shown in the figure as a dotted line through $x = a$. The Y value of the line changes from negative to positive at $x = a$. The term (c-x) is the line $y = c - x$, whose slope is -1 and x-intercept is $x = c$. The line is shown in the figure as a dotted line through $x = c$. The Y value of the line changes from positive to negative at $x = c$. The term $(x - b)^2$ is never negative so it cannot influence the sign diagram.

The SIGN of the function at the input value x, is the PRODUCT of the numbers $(x - a)(c - x)$. So put your **finger** on the number line someplace, thats your INPUT value. Now for each dotted line, ask yourself what's the value of the line at this input value? Just look to see if you have to move your finger up to meet the line $\oplus$ or go down to meet the line $\ominus$.





Now you try it for the function $f(x) = (x - 1)(x + 1)(3 - x)$. Draw a number line. Place the zero's, $-1, 1, 3$, on the number line. Draw the lines $y = x - 1$, $y = x + 1$ and $y = 3 - x$. Put your finger on the number line someplace and move vertically to meet the lines. Write the product of the outputs. Continue till your done.



Products and Ratio's of Monotone functions

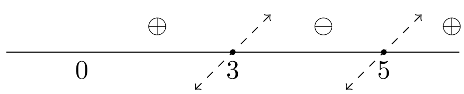
The linear terms that feature in our sign diagrams, like $(x-a)$ or $(a-x)$, are odd functions that are either increasing or decreasing. Functions that are either always increasing or always decreasing are called monotone. Anytime that we have a product or ratio of monotone functions we can create sign diagrams without any evaluations at all. For example

$$f(x) = (x-3)\sqrt[3]{x-5} \qquad \text{vs} \qquad g(x) = (x-3)(x-5)$$

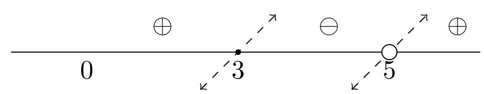
Since the cube root is an odd, increasing function, it behaves exactly like the line $y = x-5$ when constructing a sign-diagram. Both $y = x-5$, and $y = \sqrt[3]{x-5}$, are increasing. They are both zero, only at $x = 5$, hence their sign diagrams are exactly the same. Let me be clear, $f(x) \neq g(x)$, for all real numbers x, but $Sign(f(x)) = Sign(g(x))$ is. And in a sign diagram all we care about is $\oplus, \ominus$ and zero.

$$h(x) = \frac{(x-3)}{(x-5)} \quad \text{or} \quad H(x) = \frac{(x-3)}{\sqrt{x-5}}$$

ALSO, ratio's, like $h(x)$ or $H(x)$, will have the SAME sign diagram as $g(x) = (x-3)(x-5)$, EXCEPT that the functions $h(x)$ and $H(x)$ have an excluded value at $x = 5$, while $g(x)$ has a zero. The algebra of $\oplus \cdot \ominus = \frac{\oplus}{\ominus} = \frac{\ominus}{\oplus}$, works the same in a ratio as it does in a product.



Sign diagrams for $f(x)$ and $g(x)$



Sign diagrams for $h(x)$ and $H(x)$

Celebrate functions that are either always positive or always negative. For example. $f(x) = x^2 + 2$ is an example of an irreducible quadratic function; meaning that it has no real roots, and since its leading coefficient is positive its output is always positive. When you see functions like this in a sign diagram it is always gold, because it means that you can basically ignore this term. The functions

$$q(x) = (x-2)(x-5)(x^2+2) \quad \text{or} \quad Q(x) = \frac{(x-2)(x-5)}{x^2+2}$$

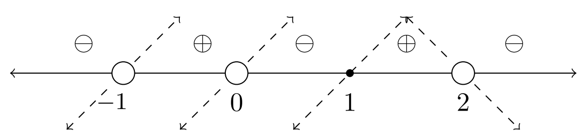
both have EXACTLY the same sign diagram as $g(x) = (x-2)(x-5)$. This suggests the following straightforward algorithm for solving sign diagrams:

1. Factor the numerator and denominator $F(x) = \frac{f_1(x)f_2(x)f_3(x)\cdots f_n(x)}{g_1(x)g_2(x)g_3(x)\cdots g_m(x)}$ into a product of monotone functions $f_i(x)$, and $g_j(x)$.
2. For the functions in the numerator mark, your number line with a solid dot for the zeros of the f_i. For the functions in the denominator mark the number line with an open circle for each excluded value.
3. For any terms that are repeated in the numerator and denominator, label the excluded value, then you can freely cancel terms according to the rules for exponents
4. For each increasing function draw a small dotted line with positive slope through its zero or excluded value
5. For each decreasing function draw a small dotted line with negative slope through its zero or excluded value
6. THINK
7. Create your sign diagram.

Lets do a seemingly complex example:

$$F(x) = \frac{(x-1)(2-x)^2(x^2+1)}{x(2-x)\sqrt[3]{x+1}}$$

1. $f_1(x) = x-1,\ f_2(x) = (2-x)^2,\ f_3(x) = x^2+2$, we label zero's at $x = 1$ only.
2. $g_1(x) = x,\ g_2(x) = 2-x,\ g_3(x) = \sqrt[3]{x+1}$, we label excluded values at $x = 0$, $x = 2$ and $x = -1$
3. We can now simplify our function $Sign(F(x)) = Sign\left\{\frac{(x-1)(2-x)}{x\sqrt[3]{x+1}}\right\}$
4. Draw a line with positive slope through $x = 0$, $x = 1$, $x = -1$
5. Draw a line with negative slope through $x = 2$.
6. Put your finger on number line to the left of $x = -1$ and compute $\ominus \cdot \ominus \cdot \ominus \cdot \oplus = \ominus$
7. Now recall the mantra: *cross a line, change the sign*, and viola your S.D. is done!



Sign diagram for $F(x) = \frac{(x-1)(2-x)^2(x^2+1)}{x(2-x)\sqrt[3]{x+1}}$