3.1 | Introduction to Rational Functions

A rational function is a ratio of polynomial functions. If $r(x)$ is a rational function then $r(x)$ is of the form:

$$r(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions.

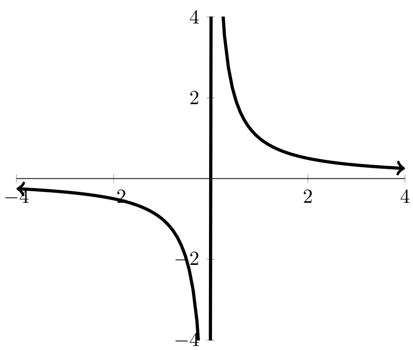
There is no one graph which can generally represent rational functions as they come in so many unique variations. However it is good to familiarize ourselves with a new type of function that will be useful throughout this section.

Laurent Monomial Functions: A Laurent monomial function is of the form

$$f(x) = \frac{a}{x^n} = ax^{-n}$$

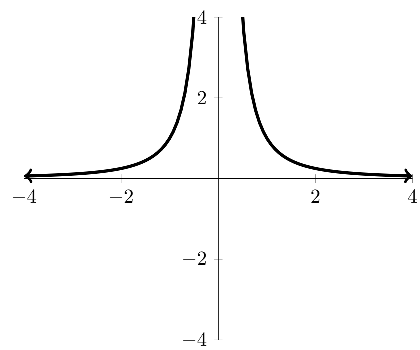
Where n is a natural number. The Laurent monomial extends the idea of monomial functions into the negative powers, and these monomial can take on unique shapes that we have not yet seen in this course.

A Laurent monomial of **odd degree** takes the form:



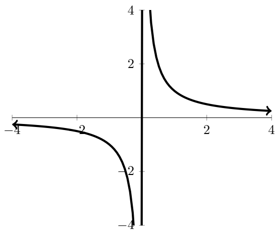
$$f(x) = \frac{1}{x}$$

A Laurent monomial of **even degree** takes the form:

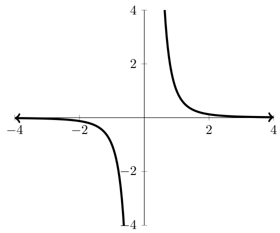


$$f(x) = \frac{1}{x^2}$$

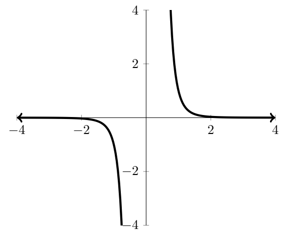
As the degree of n increases, the pattern of even and odd maintains, however the curve on the function grows sharper. An example of odd Laurent monomials of increasing degree is shown.



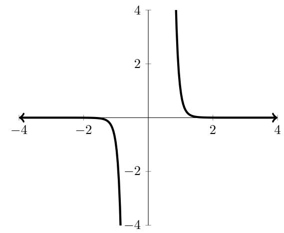
$f(x) = \frac{1}{x}$



$f(x) = \frac{1}{x^3}$



$f(x) = \frac{1}{x^5}$



$f(x) = \frac{1}{x^9}$

Asymptotes: The feature of Laurent monomials that are unique compared to functions we have seen thus far are the existence of asymptotes. Generally speaking, we can think of asymptotes as an invisible line on the graph that our function will get very close to but not touch[1]. More rigorous definitions of asymptotes are the following:

Vertical Asymptote: The line $x = c$ is called a vertical asymptote of the graph of a function $y = f(x)$ if as $x \to c^-$ or as $x \to c^+$, either $f(x) \to \infty$ or $f(x) \to -\infty$. In more general terms, a vertical asymptote $x = c$ occurs if when the x-values of the function approach c from the left or the right, then the y-values ($f(x)$) approach positive or negative infinity.

Horizontal Asymptote: The line $y = c$ is called a horizontal asymptote of the graph of a function $y = f(x)$ if as $x \to -\infty$ or $x \to \infty$, $f(x) \to c$. In more general terms, a horizontal asymptote $y = c$ occurs if when the x-values approaches positive or negative infinity, the y-values ($f(x)$) approach a finite value c.

As a general case, every Laurent monomial in the form $f(x) = ax^{-n}$ has a vertical asymptote $x = 0$ and a horizontal asymptote $y = 0$.

Transforming Functions: When we want to graph a Laurent monomial function, we can use a method known as function transformation, which looks at key values surrounding the Laurent monomial to move it around on the coordinate plane. The following theorem from the textbook goes into more detail.

[1] There are certain cases where a function can locally intersect an asymptote, however the behavior of getting close to an asymptote but not touching it occurs at values towards *infinity*.

Textbook Theorem 3.1. For real numbers a, h and k with $a \neq 0$, the graph of $F(x) = \frac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$ can be obtained from the graph of $f(x) = \frac{1}{x^n} = x^{-n}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = x^{-n}$ to $y = (x-h)^{-n}$.
   The vertical asymptote moves from $x = 0$ to $x = h$.

2. multiply the y-coordinates of the points on the graph obtained in Step 1 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = (x-h)^{-n}$ to $y = a(x-h)^{-n}$

3. add k to each of the y-coordinates of the points on the graph obtained in Step 2. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a(x-h)^{-n}$ to $y = a(x-h)^{-n} + k$.
   The horizontal asymptote moves from $y = 0$ to $y = k$.

In English 3.2. Given a function of the form $F(x) = \dfrac{a}{(x-h)^n} + k = a(x-h)^{-n} + k$, write down the parent function $P(x) = \dfrac{1}{x^n} = x^{-n}$. Write down 3 *sample points* from the parent function, then do the following:

- Add h to every x-value.
- Multiply all y-values by a.
- Add k to every y-value.

Graph these new points, then trace the parent function.

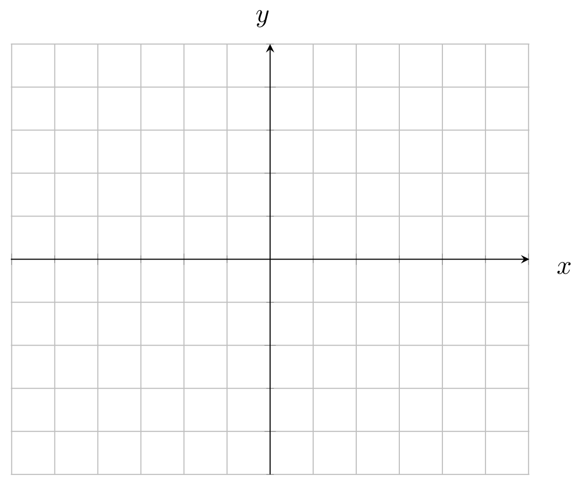
Picking Sample Points: Picking sample points from a Laurent Monomial might seem tricky at first. These functions look very unique and have asymptotes which complicate things. However, we can actually get away with using very similar sample points as from previous sections.

- **Odd Laurent Monomials:** Choose the sample points $\{(-1,-1),(0,0),(1,1)\}$.
- **Even Laurent Monomials:** Choose the sample points $\{(-1,1),(0,0),(1,1)\}$.

These choices might seem odd, as the point $(0,0)$ does not exist in any Laurent monomial functions. However, we can use $(0,0)$ as an anchor point for the intersection of the horizontal and vertical asymptotes. Just be to keep this in mind, and not plot a point at the place where $(0,0)$ ends up after using **Theorem 3.2**.

Simplifying Functions: Not all Laurent monomial functions will immediately look like $\frac{a}{(x-h)^n} + k$ and you may need to do some work in order to make it look how we want (this is necessary for properly identifying values of a, h and k). You may need to factor something out or even use synthetic division to get the function into the desired form.

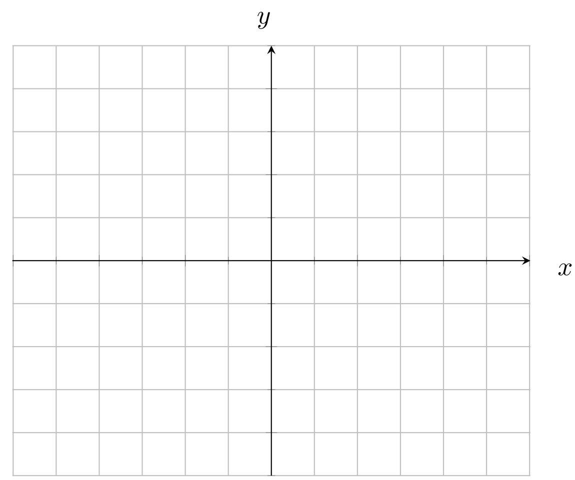
1. **Worked Example:** Sketch a graph of the function: $F(x) = \dfrac{2x-1}{x+1}$



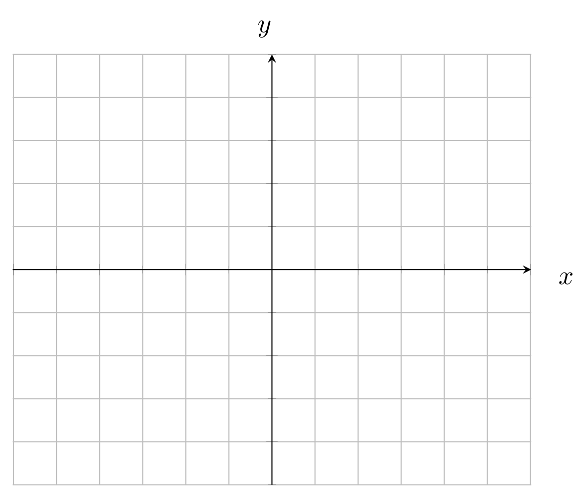
Scan the QR code for a video solution.

2. Sketch a graph of the function: $F(x) = \dfrac{1}{x-2} + 1$

3. Sketch a graph of the function: $F(x) = -(x-1)^{-2} + 3$



4. Sketch a graph of the function: $F(x) = 4x(2x+1)^{-1}$



Holes: A hole in a function is a single point which is removed from the domain. Graphically this can be represented by an open circle at the point (x, y) which corresponds to the removed x value from the domain. If c is a hole of $f(x)$ then the value $f(c)$ does not exist.

If we want to analytically verify the existence and location of asymptotes and/or holes, we have theorems that help us to do this. These theorems will of great importance later when we go to graph more complicated rational functions.

Textbook Theorem 3.3. Location of Vertical Asymptotes and Holes:[a] Suppose r is a rational function which can be written as $r(x) = \frac{p(x)}{q(x)}$ where p and q have no common zeros[b]. Let c be a real number which is not in the domain of r.

- If $q(c) \neq 0$, then the graph of $y = r(x)$ has a hole at $\left(c, \frac{p(c)}{q(c)}\right)$
- If $q(c) = 0$, then the line $x = c$ is a vertical asymptote to the graph of $y = r(x)$.

[a] Or, 'How to tell your asymptote from a hole in the graph'.
[b] In other words, $r(x)$ is in lowest terms.

In English 3.3. Location of Vertical Asymptotes and Holes: For a rational function $r(x) = \frac{p(x)}{q(x)}$, start by *factoring* both $p(x)$ and $q(x)$ (if possible). Then cancel as many terms as possible. We will let $R(x) = \frac{P(x)}{Q(x)}$ represent the new function obtained by this process of factorization and cancellation.

- If $(x - c)$ is a factor of the denominator $q(x)$ and after factoring and canceling $(x - c)$ is no longer a factor of $Q(x)$, then the zero $x = c$ will be a hole in the graph of $R(x)$ at the point $(c, R(c))$.[a]
- The zeros of any factors still present in $Q(x)$ after the process of factorization and cancellation will become vertical asymptotes on the graph of $R(x)$.

NOTE: Both holes and vertical asymptotes are values which are excluded from the domain of a rational function.

[a] If the factor was simply lowered in multiplicity through the process of cancellation, then it will not provide a hole. If the new denominator $Q(x)$ still has $x = c$ as a zero in any situation then the zero will be a asymptote of $R(x)$ not a hole.

General Process of Finding Holes and Asymptotes:

- Factor the numerator and denominator.
- Cancel any terms possible.
- Did something cancel out completely from the denominator?
  - Yes? The zeros of these factors become holes.
  - No? There are no holes in the graph.
- Set the denominator equal to zero and solve. These solutions are the vertical asymptotes of the rational function.

5. Identify any vertical asymptotes or holes of the function, if they exist: $f(x) = \dfrac{1}{x-4}$

6. Identify any vertical asymptotes or holes of the function, if they exist: $f(x) = \dfrac{1}{x^2-5x+6}$

7. Identify any vertical asymptotes or holes of the function, if they exist: $f(x) = \dfrac{x(x-2)}{x^2-5x+6}$

8. Identify any vertical asymptotes or holes of the function, if they exist: $f(x) = \dfrac{(x-1)}{(x-1)^2(x+1)}$

9. Identify any vertical asymptotes or holes of the function, if they exist: $f(x) = \dfrac{(x-1)^2}{(x-1)(x+1)}$

In addition to being able to analytically identify vertical asymptotes, we also have theorems that apply for horizontal asymptotes. Unfortunately, the theorem for horizontal asymptote includes a list of rules that will need to be memorized.

> **Textbook Theorem 3.3. Location of Horizontal Asymptotes:** Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where p and q are polynomial functions with leading coefficients a and b, respectively.
>
> - If the degree of $p(x)$ is the same as the degree of $q(x)$, then $y = \frac{a}{b}$ is the horizontal asymptote of the graph $y = r(x)$.
> - If the degree of $p(x)$ is less than the degree of $q(x)$, then $y = 0$ is the horizontal asymptote of the graph of $y = r(x)$.
> - If the degree of $p(x)$ is greater than the degree of $q(x)$, then the graph of $y = r(x)$ has no horizontal asymptotes.

10. Find the horizontal asymptote of the function, if it exists: $\dfrac{3x-2}{4x^2-4x-4}$

11. Find the horizontal asymptote of the function, if it exists: $\dfrac{3x^3-2x^2+4x+1}{x^2-x+1}$

12. Find the horizontal asymptote of the function, if it exists: $\dfrac{6x^4-2x^3+7x^2+9x-7}{3x^4-x^3+4x^2-x+5}$

In the case where the degree of $p(x)$ is greater than the degree of $q(x)$ and there is no horizontal asymptote present, this doesn't mean there is no new asymptote at all. In this case, we can have something called a **slant asymptote**. This is what the next theorem allows us to find.

Textbook Theorem 3.4. Determination of Slant Asymptotes: Suppose r is a rational function and $r(x) = \frac{p(x)}{q(x)}$, where the degree of p is exactly one more than the degree of q. Then the graph of $y = r(x)$ has the slant asymptote $y = L(x)$ where $L(x)$ is the quotient obtained by dividing $p(x)$ by $q(x)$.

In English 3.4. Determination of Slant Asymptotes: Let $r(x) = \frac{p(x)}{q(x)}$ where $p(x)$ has degree exactly one higher than that of $q(x)$. The slant asymptote is then obtained by *dividing* $p(x)$ by $q(x)$ and *ignoring the remainder*.

13. Find the slant asymptote, if it exists: $f(x) = \dfrac{x^2 - 4x + 2}{1 - x}$

14. Find the slant asymptote, if it exists: $g(t) = \dfrac{t^2 - 4}{t - 2}$

15. Find the slant asymptote, if it exists: $h(x) = \dfrac{x^3 + 1}{x^2 - 4}$