

## 5.2 | Function Arithmetic

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In this section we introduce definitions and theorems that generalize the way in which we can treat functions as their own objects and do arithmetic with them. Many of these definitions/theorems turn out exactly as we would expect, but it is still useful to write them down as a reference point in what we are allowed to do with functions.

**Definitions of Arithmetic:** Suppose  $f$  and  $g$  are functions and  $x$  is in both the domain of  $f$  and the domain of  $g$ .

- **Sum:**  $f + g$  is defined by the formula:

$$(f + g)(x) = f(x) + g(x)$$

- **Difference:**  $f - g$  is defined by the formula:

$$(f - g)(x) = f(x) - g(x)$$

- **Product:**  $fg$  is defined by the formula:

$$(fg)(x) = f(x)g(x)$$

- **Quotient:**  $\frac{f}{g}$  is defined by the formula:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

provided that  $g(x) \neq 0$ .

In addition to the basic arithmetic operations, the arithmetic rules follow as outlined in the following theorem.

**Textbook Theorem 5.1.** Suppose  $f$ ,  $g$ , and  $h$  are functions.

- **Commutative Law of Addition:**  $f + g = g + f$
- **Associative Law of Addition:**  $(f + g) + h = f + (g + h)$
- **Additive Identity:** The function  $Z(x) = 0$  satisfies:  $f + Z = Z + f = f$  for all functions  $f$ .
- **Additive Inverse:** The function  $F(x) = -f(x)$  for all  $x$  in the domain of  $f$  satisfies:

$$f + F = F + f = Z$$

- **Commutative Law of Multiplication:**  $fg = gf$
- **Associative Law of Multiplication:**  $(fg)h = f(gh)$
- **Multiplicative Identity:** The function  $I(x) = 1$  satisfies  $fI = If = f$  for all functions  $f$ .
- **Multiplicative Inverse:** If  $f(x) \neq 0$  for all  $x$  in the domain of  $f$ , then  $F(x) = \frac{1}{f(x)}$  satisfies:

$$fF = Ff = I$$

- **Distributive Law of Multiplication over Addition:**  $f(g + h) = fg + fh$

1. Given  $f(x) = 3x + 1$  and  $g(t) = 4 - t$  compute the following:

(a)  $(f + g)(2)$

(b)  $(f - g)(-1)$

2. Given  $f(x) = x^2$  and  $g(t) = -2t + 1$  compute the following:

(a)  $(g - f)(1)$

(b)  $(fg)\left(\frac{1}{2}\right)$

3. Given  $f(x) = x^2 - x$  and  $g(t) = 12 - t^2$  compute the following:

(a)  $\left(\frac{f}{g}\right)(0)$

(b)  $\left(\frac{g}{f}\right)(-2)$

4. Let  $f$  be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let  $g$  be the function defined by

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

compute the indicated value if it exists:

(a)  $(f - g)(2)$

(b)  $(gf)(-3)$

(c)  $(g + f)(1)$

(d)  $\left(\frac{f}{g}\right)(-1)$

**Difference Quotient:** A difference quotient<sup>1</sup> is a formula that helps us measure the amount of change in a function at a given point. A common formula for difference quotients is:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

with  $h \neq 0$ . Here, we read the Greek character  $\Delta$  as “change.” So  $\Delta[f(x)]$  can be interpreted as “change in  $f(x)$ .” We measure this change by adding a value to  $x$  often called  $h$ . So  $x+h$  is a new value which differs from  $x$  by  $h$  amount. Then we compute the *function value* at this point  $f(x+h)$ , and then we compute the function value at the original point (just  $x$ )  $f(x)$ . We can then measure the average amount of change between these points by subtracting  $f(x)$  from  $f(x+h)$  and then averaging over  $h$ . Putting this altogether yields the formula stated above.<sup>2</sup>

There are variations on the difference quotient formula. For one, sometimes we will be provided the value of  $x$  in advance. For example if we want to compute when  $x = 2$  our formula would look like:

$$\frac{f(2+h) - f(2)}{h}$$

Alternatively, some formulas write  $h$  as  $\Delta x$  to emphasize that  $h$  is meant to represent an amount of change in  $x$ . You can treat “ $\Delta x$ ” as a separate variable entirely from  $x$  (as opposed to reading it as  $\Delta$  *times*  $x$ ). This formula would look like:

$$\frac{f(x + \Delta x) - f(x)}{\Delta x}$$

**Solving Difference Quotients:** To solve a difference quotient, we will simply follow the formula provided. If you are given a function  $f$  and you are trying to compute  $f(x+h)$ , replace every instance of  $x$  (or another variable used) in the function  $f$ , with  $(x+h)$ . Then simplify the new function. For example: if  $f(x) = x + 3$ , then  $f(x+h) = (x+h) + 3 = x+h+3$ .

**Problem Solving Tip 1.** When solving a difference quotient, the algebra can get very lengthy and keeping everything inside a fraction can be difficult. It may be helpful to *separately* write out the definition of  $f(x+h)$  and *fully simplify it* before writing it in the formula.

**Problem Solving Tip 2.** In the formula for difference quotients, be very careful with your minus sign in between  $f(x+h)$  and  $f(x)$ . It can help to write the expanded form of  $f(x)$  in large brackets to remind yourself that the minus sign needs to be fully distributed throughout the entire function  $f(x)$ .

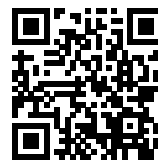
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<sup>1</sup>Sometimes called a Newton Quotient.

<sup>2</sup>Difference quotients as a formula might not make a lot of sense in the context of this course. The skill of finding difference quotients is one that will make a return during the study of calculus. A difference quotient can be used to measure the *instantaneous rate of change* by analyzing what happens as the value of  $h$  grows smaller and smaller, getting as close to 0 as possible. This can be visualized as the *slope of a curve at a point*. For those who are more interested in this phenomenon, please see the following YouTube video [The paradox of the derivative](#) | Chapter 2, Essence of calculus.

For the following practice problems, use the difference quotient formula:  $\frac{f(x+h) - f(x)}{h}$ .

5. **Worked Example:** Compute the difference quotient:  $f(x) = 2x - 5$



Scan the QR code for a video solution.

6. Compute the difference quotient:  $f(x) = -3x + 5$

7. Compute the difference quotient:  $f(x) = 6$

8. Compute the difference quotient:  $f(x) = 3x^2 - x$

9. Compute the difference quotient:  $f(x) = x - x^2$

10. Compute the difference quotient:  $f(x) = ax^2 + bx + c$  where  $a \neq 0$ .

Materials in PAL are not a suitable replacement for materials in class. These materials are not for use on exams.