5.4 | Transformations of Graphs

In previous sections, we saw methods in which we generalized families of functions into a general form, and then traced key points from a parent function onto a new graph. In this section, we generalize this idea to *any* function.

The following two theorems highlight processes we have seen before. We will still include them for continuity sake.

Textbook Theorem 5.3. Vertical Shifts. Suppose f is a function and k is a real number.
To graph $F(x) = f(x) + k$, add k to each of the y-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a vertical shift up k units if $k > 0$ or down k units if $k < 0$.

Textbook Theorem 5.4. Horizontal Shifts. Suppose f is a function and h is a real number.
To graph $F(x) = f(x - h)$, add h to each of the x-coordinates of the points on the graph of $y = f(x)$.
NOTE: This results in a horizontal shift right h units if $h > 0$ or left if $h < 0$.

These next three theorems elaborate on processes we might be intuitively familiar with, but now we finally state them as a theorem.

Textbook Theorem 5.5. Reflections. Suppose f is a function.
To graph $F(x) = -f(x)$, multiply each of the y-coordinates of the points on the graph by $y = f(x)$ by -1.
NOTE: This results in a reflections across the x-axis
To graph $F(x) = f(-x)$, multiply each of the x-coordinates of the points on the graph of $y = f(x)$ by -1.
NOTE: This results in a reflection across the y-axis.

Textbook Theorem 5.6. Vertical Scalings. Suppose f is a function and $a > 0$ is a real number.
To graph $F(x) = af(x)$, multiply each of the y-coordinates of the points on the graph $y = f(x)$ by a.

- If $a > 1$, we say the graph of f has undergone a vertical stretch[a] by a factor of a.
- If $0 < a < 1$, we say that the graph of f has undergone a vertical shrink[b] by a factor of $\frac{1}{a}$.

[a] expansion, dilation
[b] compression, contraction

Textbook Theorem 5.7.

Horizontal Scalings. Suppose f is a function and $b > 0$ is a real number.
To graph $F(x) = f(bx)$, divide each of the x-coordinates of the points on the graph of $y = f(x)$ by b.

- If $0 < b < 1$, we say the graph of f as undergone a horizontal stretch[a] by a factor of b.
- If $b > 1$ we say the graph of f has undergone a horizontal shrink[b] by a factor of $\frac{1}{b}$.

[a] expansion, dilation
[b] compression, contraction

Finally, we can put all these theorems together and we get something that looks very familiar to theorems that we have seen in previous sections. Now, we have the ability to generalize this to any function we like.

Textbook Theorem 5.8. Transformations in Sequence. Suppose f is a function. If $a, b \neq 0$, then to graph $g(x) = af(bx - h) + k$ start with the graph of $y = f(x)$ and follow the steps below.

1. Add h to each of the x-coordinates of the points on the graph of f.
   NOTE: This results in a horizontal shift to the left if $h < 0$ or right if $h > 0$.
2. Divide the x-coordinates of the points on the graph obtained in Step 1 by b.
   NOTE: This results in a horizontal scaling, but includes a reflection about the y-axis if $b < 0$.
3. Multiply the y-coordinates of the points on the graph obtained in Step 2 by a.
   NOTE: This results in a vertical scaling, but includes a reflection about the x-axis if $a < 0$.
4. Add k to each of the y-coordinates of the points on the graph obtained in Step 3.
   NOTE: This results in a vertical shift up if $k > 0$ or down if $k < 0$.

In English 5.8. Transformations in Sequence. If $f(x)$ is a function. We can graph $F(x) = a \cdot f(bx - h) + k$ using the following method on points (x, y) taken from $f(x)$. Be sure to do these steps in order.

1. Add h to every x-coordinate.
2. Divide these new x-coordinates by b.
3. Multiply every y coordinate by a.
4. Add k to these new y-coordinates.

The resulting pair of (x, y) coordinates will exist on the function $F(x) = a \cdot f(bx - h) + k$.

For the following problems, suppose $(2, -3)$ is on the graph $y = f(x)$. Use theorem 5.8 to find a point on the graph of the given function.

1. **Worked Example:** $y = f(x - 3) + 1$

Scan the QR code for a video solution

2. $y = 2f(x + 1)$

3. $y = 3f(2x) - 1$

4. $y = \frac{1}{2}f(4 - x)$

For the following problems, suppose $(2, -3)$ is on the graph $y = f(x)$. Use theorem 5.8 to find a point on the graph of the given function.

5. $y = 5f(2x + 1) + 3$

6. $y = 2f(1 - x) - 1$

7. $y = f\left(\dfrac{7 - 2x}{4}\right)$

8. $y = \dfrac{f(3x) - 1}{2}$