MATH1300
Selected Challenge Problems
Volume II

SOLUTIONS

Precalculus Peer Assisted Learning

October 22, 2024

Solution Preface:

I make no guarantee that these solutions are free of typos or errors. I have done my best to ensure they are all correct; however, you should always double-check my solutions with another resource if you are unsure whether they are accurate.

Function graphs in this packet are generated by a computer and may not look exactly like the ones you draw. Remember, you are not a computer. The most important skill to develop is your ability to analyze key information about the graph and use a sign diagram to sketch a relatively accurate picture.

Roman

2.2

A Let $f(z) = 4z^3 + 2z - 3$ and $g(z) = z - 3$

i. Compute $f(z)/g(z)$.
$(4z^3 + 2z - 3) \div (z - 3) = (4x^2 + 12z + 38)\ R11$

ii. Write $f(z)$ as an expression involving $g(z)$, a quotient, and remainder (if it exists).
$(4z^3 + 2z - 3) = (z - 3)(4z^2 + 12z + 38) + 11$

B Let $f(x) = 2x^3 - x + 1$ and $g(x) = x^2 + x + 1$

i. Compute $f(x)/g(x)$.
$(2x^3 - x + 1) \div (x^2 + x + 1) = (2x - 2)\ R(3 - x)$

ii. Write $f(x)$ as an expression involving $g(x)$, a quotient, and remainder (if it exists).
$(2x^3 - x + 1) = (2x - 2)(x^2 + x + 1) + (3 - x)$

C Let $a(x) = x^4 - 6x^2 + 9$ and $b(x) = (x - \sqrt{3})$

i. Compute $a(x)/b(x)$.
$(x^4 - 6x^2 + 9) \div (x - \sqrt{3}) = (x^3 + \sqrt{3}x^2 - 3x - 3\sqrt{3})\ R0$

ii. Write $a(x)$ as an expression involving $b(x)$, a quotient, and remainder (if it exists).
$x^4 - 6x^2 + 9 = (x^3 + \sqrt{3}x^2 - 3x - 3\sqrt{3})(x - \sqrt{3})$

D Let $g(z) = z^3 + 2z^2 - 3z - 6$ be a polynomial function with a known real zero of $c = -2$

i. Find the remaining real zeros of $g(z)$
$z = -2, \sqrt{3}, -\sqrt{3}$

E Let $x^3 - 6x^2 + 11x - 6$ be a polynomial function with a known real zero of $c = 1$

i. Find the remaining real zeros of $g(z)$
$x = 1, 2, 3$

F* Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with the property that $a_n + a_{n-1} + \cdots + a_1 + a_0 = 0$. (That is, the sum of the coefficients and the constant term is 0.)

i. Show that $(x - 1)$ is a factor of $f(x)$.

Proof.

If $(x - 1)$ is a factor then $f(1) = 0$. Plug in $x = 1$ to $f(x)$ to obtain $f(1) = (a_n)1 + (a_{n-1})1 + \cdots + (a_1)1 + (a_0)1$ which simplifies to $a_n + a_{n-1} + \cdots + a_1 + a_0$. By our assumption, $a_n + a_{n-1} + \cdots + a_1 + a_0 = 0$. So $f(1) = 0$ and thus $(x - 1)$ is a factor of f. □

2.3

A Let $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

i. Use Cauchy's Bound to find an interval containing all possible rational zeros.
$\left[-\frac{4}{3}, \frac{4}{3}\right]$

ii. Use the Rational Zeros Theorem to make a list of possible rational zeros.
$\left\{\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{1}{3}, \pm\frac{1}{4}, \pm\frac{1}{6}, \pm\frac{1}{9}, \pm\frac{1}{12}, \pm\frac{1}{18}, \pm\frac{1}{36},\right\}$

iii. Use Descartes' Rule of Signs to list the possible number of positive and negative real zeros.
2 or 0 positive real zeros, 2 or 0 negative real zeros.

B Let $p(z) = 2z^4 + z^3 - 7z^2 - 3z + 3$

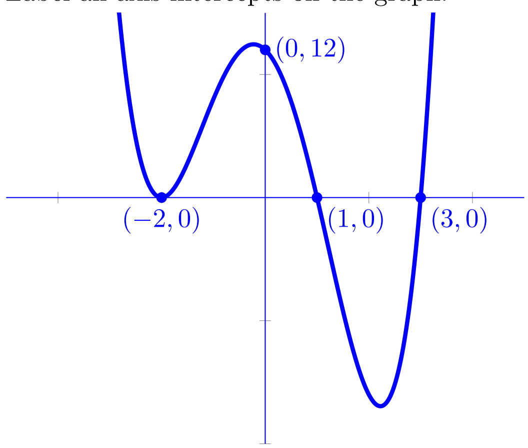
i. Use the Rational Zeros Theorem to list possible roots of the polynomial.
$\left\{\pm 3, \pm 1, \pm\frac{3}{2}, \pm\frac{1}{2}\right\}$

ii. Write the polynomial in factored form.
$(2z - 1)(z + 1)(z^2 - 3)$

C Let $g(x) = x^4 - 9x^2 - 4x + 12$

i. Sketch the graph of $g(x)$.

ii. Label all axis intercepts on the graph.



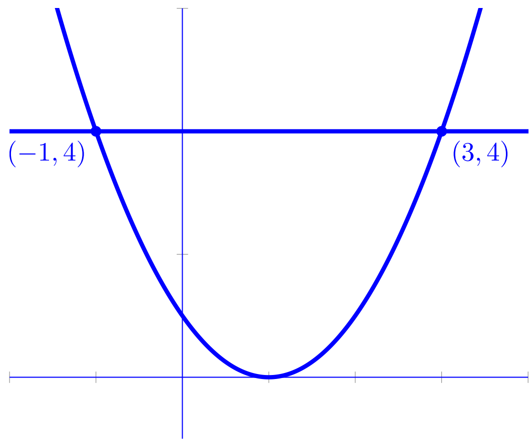
iii. Write the end behavior of $g(x)$.
as $x \to -\infty$, $f(x) \to \infty$
as $x \to \infty$, $f(x) \to \infty$

D Solve the following equation: $x^3 + x^2 = \frac{11x + 10}{3}$

$x = -2, \frac{3 \pm \sqrt{69}}{6}$

E Let $f(x) = (x-1)^2$ and $g(x) = 4$

i. Graph $f(x)$ and $g(x)$ on the same coordinate plane.



ii. Solve the inequality $f(x) \geq g(x)$ graphically.
$(-\infty, -1] \cup [3, \infty)$

iii. Solve the inequality $f(x) \geq g(x)$ algebraically and verify that it matches the solution found in part (ii).
$(-\infty, -1] \cup [3, \infty)$

F Solve the inequality: $\dfrac{x^3 + 20x}{8} \geq x^2 + 2$, express your answer in interval notation.
$\{2\} \cup [4, \infty)$

3.1

A Let $p(x) = 9x^3 + 5$ and $q(x) = 2x - 3$

i. Divide $p(x) \div q(x)$ using synthetic division or long division.
Synthetic division will make dealing with the fractions in this problem easier.

ii. Write $p(x)$ in the form of $d(x)q(x) + r(x)$.
$(9x^3 + 5) = (2x - 3)\left(\frac{9}{2}x^2 + \frac{27}{4}x + \frac{81}{8}\right) + \frac{283}{8}$

B Let $p(x) = 4x^2 - x - 23$ and $q(x) = x^2 - 1$

i. Divide $p(x) \div q(x)$ using synthetic division or long division.
Must use long division as synthetic division will not work for non linear divisors.

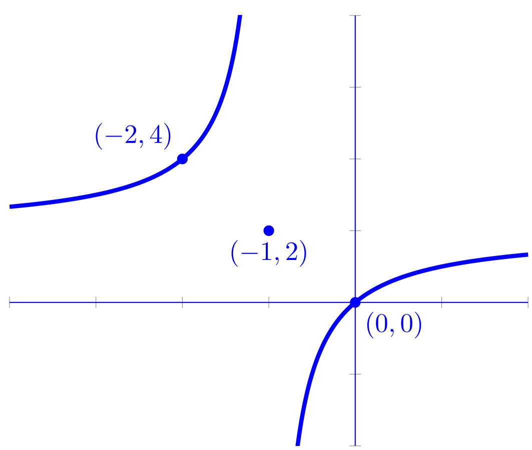
ii. Write $p(x)$ in the form of $d(x)q(x) + r(x)$.
$4x^2 - x - 23 = 4(x^2 - 1) + (-x - 19)$

C Let $h(x) = \dfrac{2x}{x+1}$.

i. Write $h(x)$ in the form $\dfrac{a}{x-h} + k$.
Use division to obtain $h(x) = 2 - \dfrac{2}{x+1}$

ii. Write the parent function $P(x)$ of $h(x)$.
$P(x) = \dfrac{1}{x}$

iii. Track at least two points and the asymptotes from $P(x)$ and use them to graph $h(x)$.
Choose sample points $\{(-1,-1), (1,1)\}$ and track $(0,0)$ for asymptotes.



D Let $r(x) = \dfrac{x^2 - x - 12}{x^2 + x - 6}$

i. Identify any holes in the graph of $r(x)$.
$\left(-3, \frac{7}{5}\right)$

ii. Identify any vertical asymptotes in the graph of $r(x)$.
$x = 2$

iii. State the domain of $r(x)$.
$(-\infty, -3) \cup (-3, 2) \cup (2, \infty)$

E Let $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2}$

i. Identify any holes in the graph of $f(x)$.
$(-1, 0)$

ii. Identify any vertical asymptotes in the graph of $f(x)$.
$x = 2$

iii. State the domain of $f(x)$.
$(-\infty, -1) \cup (-1, 2) \cup (2, \infty)$

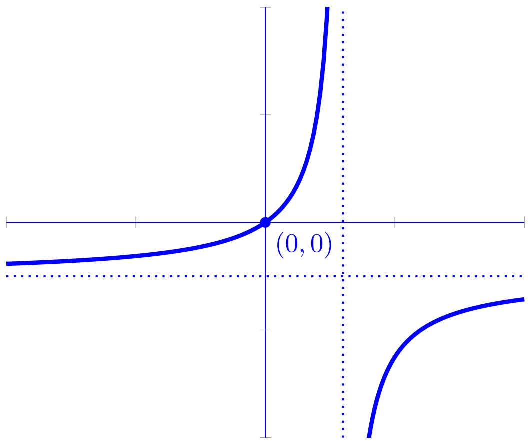
F* Let $u(x)$ be a function defined only on the positive real numbers. Let $v(x) = (x - a)(x + b)$ with $0 < a < b$.

i. State the domain of $w(x) = \dfrac{u(x)}{v(x)}$
$(0, a) \cup (a, \infty)$

3.2

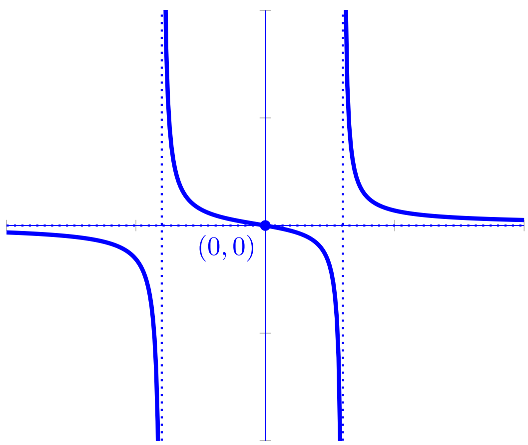
A Let $f(x) = 5x(6-2x)^{-1}$

i. Sketch the graph of $f(x)$. Label all asymptotes, holes, and zeros.



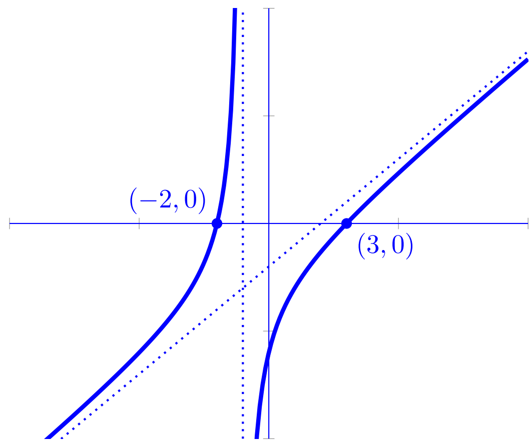
B Let $a(x) = \dfrac{x}{x^2+x-12}$

i. Sketch the graph of $a(x)$. Label all asymptotes, holes, and zeros.



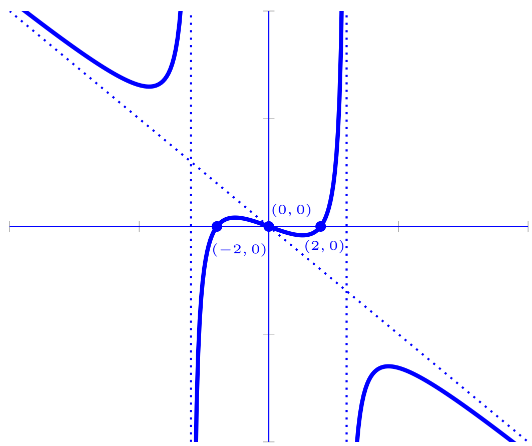
C Let $r(t) = \dfrac{t^2-t-6}{t+1}$

i. Sketch the graph of $r(t)$. Label all asymptotes, holes, and zeros.



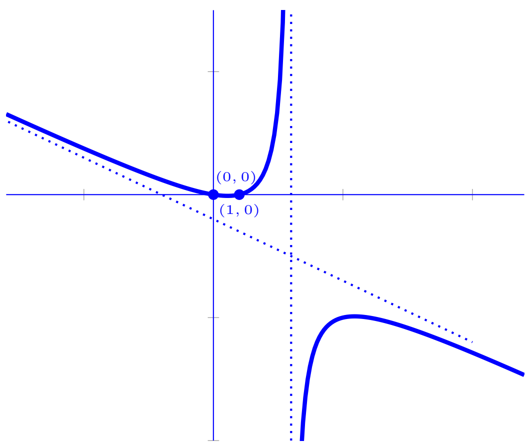
D Let $f(x) = \dfrac{5x}{9 - x^2} - x$

i. Sketch the graph of $f(x)$. Label all asymptotes, holes, and zeros.



E Let $r(z) = -z - 2 + \dfrac{6}{3-z}$

i. Sketch the graph of $r(z)$. Label all asymptotes, holes, and zeros.



F* Let $p(x) = 2x^3 + 5x^2 + 4x + 3$ and $q(x) = 2x + 1$

i. Does $r(x) = \dfrac{p(x)}{q(x)}$ have a horizontal or slant asymptote?
Neither.

ii. Divide $p(x) \div q(x)$ and ignore the remainder. What does this suggest about the (non vertical) asymptote of $r(x)$?
Dividing and ignoring the remainder obtains: $x^2 + 2x + 1$. This suggests that the asymptote of $r(x)$ is a *parabola*.

iii. Assume $a(x)$ is a fourth degree polynomial, and $b(x)$ is a linear. Assuming $b(x)$ is not a factor of $a(x)$, what might the (non vertical) asymptote of $f(x) = \dfrac{a(x)}{b(x)}$ look like?
Dividing a degree four polynomial by a linear term yields a third degree polynomial. So the asymptote of $f(x)$ would be a *cubic function*.

3.3

A Solve $\dfrac{3x-1}{x^2+1} = 1$.
$x = 1, 2$

B Solve $\dfrac{1}{t+3} + \dfrac{1}{t-3} = \dfrac{t^2-3}{t^2-9}$.
$t = -1$

C Solve $\dfrac{4t}{t^2+4} \geq 0$.
$[0, \infty)$

D Solve $\dfrac{2t+6}{t^2+t-6} < 1$.
$(-\infty, -3) \cup (-3, 2) \cup (4, \infty)$

E Solve $\dfrac{3z-1}{z^2+1} \leq 1$.
$(-\infty, 1] \cup [2, \infty)$

F* Solve $\dfrac{2x^2-5x+4}{3x^2+1} < 0$, justify your answer.
$3x^2+1$ is always positive, use the discriminant and vertex form to show that $2x^2-5x+4$ is also always positive, so there are no solutions.

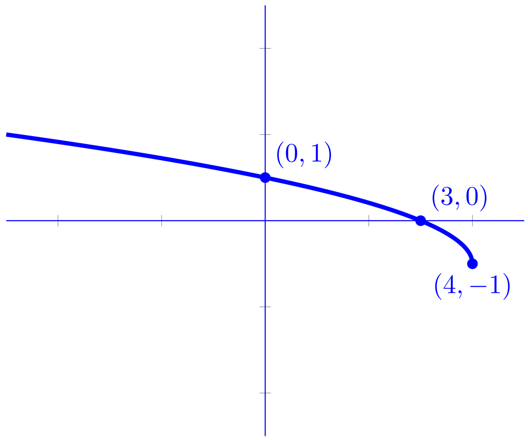
4.1

A Let $f(x) = \sqrt{4-x} - 1$

i. Write the parent function $P(x)$ for f.
$P(x) = \sqrt{x}$

ii. Track at least three points from $P(x)$ and use them to graph $f(x)$.

Track $(0,0), (1,1)$, and $(4,2)$

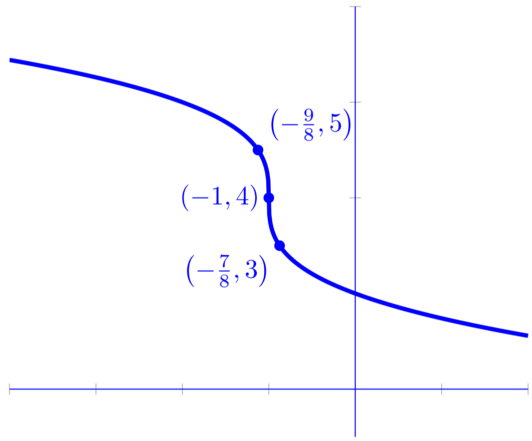


B Let $f(x) = -\sqrt[3]{8x+8} + 4$

i. Write the parent function $P(x)$ for f.
   $P(x) = \sqrt[3]{x}$

ii. Track at least three points from $P(x)$ and use them to graph $f(x)$.

   Track $(-1,-1), (0,0)$, and $(1,1)$

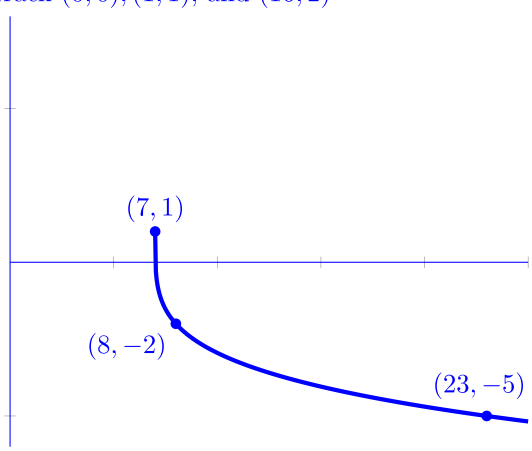


C Let $f(x) = -3\sqrt[4]{x-7} + 1$

i. Write the parent function $P(x)$ for f.
   $P(x) = \sqrt[4]{x}$

ii. Track at least three points from $P(x)$ and use them to graph $f(x)$.

   Track $(0,0), (1,1)$, and $(16,2)$



D Let $d(x) = \dfrac{5x}{\sqrt[3]{x^3+8}}$

i. State the domain of $d(x)$.
$(-\infty, -2) \cup (-2, \infty)$

E Let $z(x) = \sqrt{x(x+5)(x-4)}$

i. State the domain of $z(x)$.
$[-5, 0] \cup [4, \infty)$

F Let $c(x) = \sqrt[6]{\dfrac{x^2+x-6}{x^2-2x-15}}$

i. State the domain of $c(x)$.
$(-\infty, -3) \cup (-3, 2] \cup (5, \infty)$

4.2

A Let $c(x) = x^{\frac{4}{7}}$

i. List the intervals where $c(x)$ is increasing (if any exist).
$(0, \infty)$

ii. List the intervals where $c(x)$ is decreasing (if any exist).
$(-\infty, 0)$

iii. List the intervals where $c(x)$ is concave up (if any exist).
No intervals exist.

iv. List the intervals where $c(x)$ is concave down (if any exist).
$(-\infty, 0) \cup (0, \infty)$

B Let $b(t) = t^{\frac{10}{4}}$

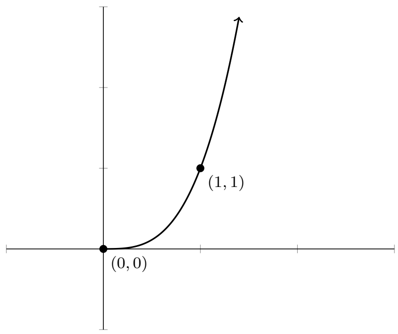
i. List the intervals where $c(x)$ is increasing (if any exist).
$(0, \infty)$

ii. List the intervals where $c(x)$ is decreasing (if any exist).
No intervals exist.

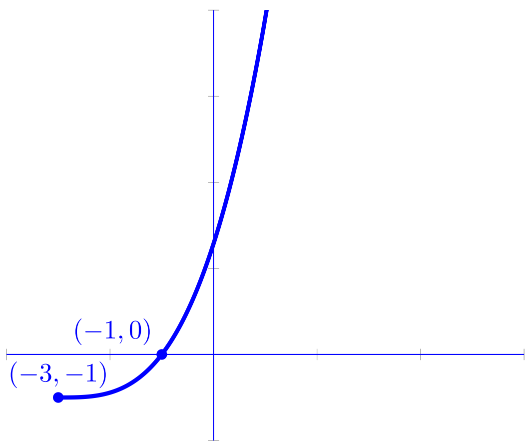
iii. List the intervals where $c(x)$ is concave up (if any exist).
$(0, \infty)$

iv. List the intervals where $c(x)$ is concave down (if any exist).
No intervals exist.

C The graph $g(t) = t^{\pi}$ is shown (where $\pi \approx 3.1415\ldots$).



i. Track the points provided on $g(t)$ to graph $G(t) = \left(\frac{t+3}{2}\right)^{\pi} - 1$



D Let $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$

i. State the domain of $f(x)$.
$[0, \infty)$

E Let $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$

i. State the domain of $f(x)$.
$(2, \infty)$

F* Let $g(t) = 4t(9-t^2)^{-\sqrt{2}}$

i. State the domain of $g(t)$.
$(-3, 3)$

4.3

A Solve the equation $2x + 1 = (3 - 3x)^{\frac{1}{2}}$
$x = \frac{1}{4}$

B Solve the equation $(2x + 1)^{\frac{1}{2}} = 3 + (4 - x)^{\frac{1}{2}}$
$x = 4$

C Solve the equation $2t^{\frac{1}{3}} = 1 - 3t^{\frac{2}{3}}$
$t = -1, \frac{1}{27}$

D Solve the inequality $\sqrt[3]{x} > x$, express your answer in interval notation.
$(-\infty, -1) \cup (0, 1)$

E Solve the inequality $(2 - 3x)^{\frac{1}{3}} > 3x$, express your answer in interval notation.
$\left(-\infty, \frac{1}{3}\right)$

F Solve the inequality $3(x - 1)^{\frac{1}{3}} + x(x - 1)^{-\frac{2}{3}} \geq 0$, express your answer in interval notation.
$\left[\frac{3}{4}, 1\right) \cup (1, \infty)$