6.2 | Logarithmic Functions

Logarithmic Functions: A logarithmic function is of the form

$$f(x) = \log_b(x)$$

where b is a real number, $b > 0$ and $b \neq 1$. We refer to the number b as the base. Logarithmic functions have the very important property of being the inverse of an exponential function. That is, if $f(x) = b^x$, then $f^{-1}(x) = \log_b(x)$.

Shorthand Notation: Some logarithms are common enough that we reserve special names for them. This is similar to how if we write $\sqrt{x}$ it is implies that we are taking the square root $\sqrt[2]{x}$.

- **Common Logarithm**: The common logarithm is a logarithm of base 10. Base 10 is very common in scientific notation and shows up often fields outside of mathematics.[1] The shorthand for this is:

$$\log_{10}(x) = \log(x) \qquad \text{(Common Log)}$$

- **Natural Logarithm:** The natural logarithm is of base e. Base e shows up most often in pure mathematical contexts, but can also appear in many other fields as well. The shorthand for this is:

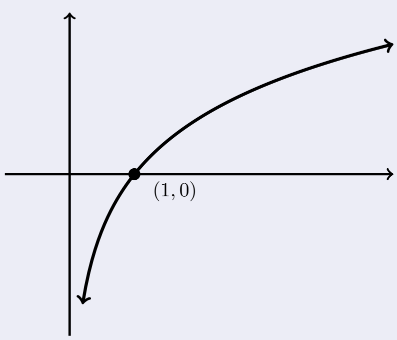
$$\log_e x = \ln(x) \qquad \text{(Natural Log)}$$

[1] In the United States, $\log(x)$ typically implies a base of 10, especially in school settings. However, in other countries and certain scientific fields, $\log(x)$ may imply different bases. For example, mathematicians outside the U.S. often denote $\log(x)$ as the natural logarithm (base e), while computer scientists frequently use $\log(x)$ to represent a logarithm with base 2. When reading scientific papers, be sure to check what $\log(x)$ is being used to represent.

Textbook Theorem 6.3. Properties of Exponential Functions: Suppose $f(x) = b^x$.

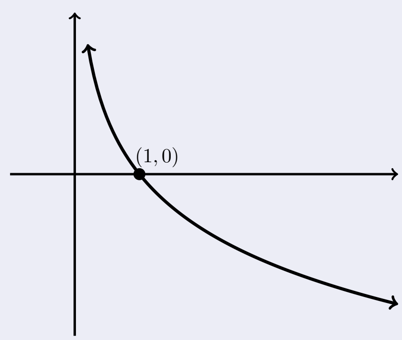
- The domain of f is $(0, \infty)$ and the range of f is $(-\infty, \infty)$.
- $(1, 0)$ is on the graph of f and $x = 0$ is a vertical asymptote to the graph of f.
- f is one-to-one, continuous, and smooth.
- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b(b^x) = x$ for all real number x and $b^{\log_b(x)} = x$ for all $x > 0$.

- If $b > 1$:
  - f is always increasing
  - As $x \to 0^+$, $f(x) \to -\infty$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = \log_b(x), b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to 0^+$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to -\infty$
  - The graph of f resembles:



$y = \log_b(x), 0 < b < 1$

Rewriting Logarithms: Theorem 6.3 contains a lot of information, but a key takeaway is the following idea:

$$b^a = c \text{ if and only if } \log_b(c) = a$$

Some students like to visualize this by imagining the base b swinging over into an exponential equation and dropping the logarithm.[2]

$$\log_b(c) = a \longrightarrow c = b^a$$

[2] I recommend not to rely on this too heavily. It is more mathematically sound to understand how logarithmic functions are inverse of exponential. An idea which we will explore more in depth in section 6.3.

1. Rewrite the logarithmic expression as an exponential expression: $\log_5(25) = 2$

2. Rewrite the logarithmic expression as an exponential expression: $\log_{25}(5) = \frac{1}{2}$

3. Rewrite the logarithmic expression as an exponential expression: $\ln(e) = 1$

4. Rewrite the exponential expression as a logarithmic expression: $5^{-3} = \frac{1}{125}$

5. Rewrite the exponential expression as a logarithmic expression: $4^{5/2} = 32$

6. Rewrite the exponential expression as a logarithmic expression: $e^0 = 1$

Problem Solving Tip 1. Evaluating Logarithms: If you want to evaluate a logarithmic expression outside of an equation. Simply set the logarithm to an unknown (for example, x), and solve for the unknown while utilizing the property:

$$b^a = c \text{ if and only if } \log_b(c) = a$$

7. Evaluate the expression without using a calculator: $\log_3(27)$

8. Evaluate the expression without using a calculator: $\log_2(32)$

9. Evaluate the expression without using a calculator: $\ln(e^3)$

10. Evaluate the expression without using a calculator: $\log_4(8)$

Finding the Domain of Logarithms: To find the domain of a logarithm, set the expression inside the logarithm greater to or equal to zero and solve. These solutions will be the domain of the logarithm.

11. Find the domain: $f(x) = \log\left(\frac{x+2}{x^2-1}\right)$

12. Find the domain: $f(x) = \log\left(\frac{x^2+9x+18}{4x-20}\right)$

13. Find the domain[3]: $g(t) = \ln(7-t) + \ln(t-4)$

Materials in PAL are not a suitable replacement for materials in class. These materials are not for use on exams.

[3] Logarithm simplification rules are not introduced until section 6.3. Can you find a way to do this problem without combining the logarithms?