6.3 | Properties of Logarithms

The following set of theorems are very important to the study of logarithms.

Textbook Theorem 6.4. (Inverse Properties of Exponential and Logarithmic Functions)
Let $b > 0$, $b \neq 1$.

- $b^a = c$ if and only if $\log_b(c) = a$. That is, $\log_b(c)$ is the exponent you put on b to obtain c.
- $\log_b(b^x) = x$ for all x and $b^{\log_b(x)} = x$ for all $x > 0$.

In English 6.4. (Inverse Properties of Exponential and Logarithmic Functions)
Let $b > 0$, $b \neq 1$.

- If $b^a = c$ then $\log_b(c) = a$ and if $\log_b(c) = a$ then $b^a = c$.
- $\log_b(b^x) = x = b^{\log_b(x)}$ for all $x > 0$.

The property $\log_b(b^x) = x = b^{\log_b(x)}$ is an extremely important one that will serve as the foundation for solving equations with both exponents and logarithms. What this equation tells us, is that in the same way that we undid squares with square roots, and powers of n with nth roots, we can **undo exponents using logarithms and undo logarithms using exponents**, all we need to do is choose the appropriate base. This property will be utilized much more in sections 6.4 and 6.5.

Problem Solving Tip 1. Get a tattoo of $\log_b(b^x) = x = b^{\log_b(x)}$

The following theorem is an application of the property $\log_b(b^x) = x = b^{\log_b(x)}$.

Textbook Theorem 6.5. (One-to-one Properties of Exponential and Logarithmic Functions)
Let $f(x) = b^x$ and $g(x) = \log_b(x)$ where $b > 0$, $b \neq 1$. Then f and g are one-to-one and

- $b^u = b^w$ if and only if $u = w$ for all real numbers u and w.
- $\log_b(u) = \log_b(w)$ if and only if $u = w$ for all reals numbers $u > 0$, $w > 0$.

This theorem feels trivial, and in many ways it is, but it can be valuable to reflect on *why* theorem 6.5 is true. If we are given an equation $b^u = b^w$, we can perform any operation as long as anything we do to one side we do to the other. In this case, we take the log base b of both sides, to obtain $\log_b(b^u) = \log_b(b^w)$. Then we can apply the property $\log_b(b^x) = x = b^{\log_b(x)}$ to simplify and find that $u = w$. The argument as to why $\log_b(u) = \log_b(w)$ implies $u = w$ is similar.

The following theorem is theorem 6.2 from worksheet 6.1, but it gets restated for convenience.

Textbook Theorem 6.6. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function $(b > 0, b \neq 1)$ and let u and w be real numbers.

- **Product Rule:** $f(u+w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u-w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

In English 6.6. (Algebraic Properties of Exponents) For any base b and real numbers x and y:

- $b^x b^y = b^{x+y}$
- $\dfrac{b^x}{b^y} = b^{x-y}$
- $(b^x)^y = b^{xy}$

Now we introduce the analogous theorem for logarithmic functions.

Textbook Theorem 6.7. (Algebraic Properties of Logarithmic Functions) Let $g(x) = \log_b(x)$ be a logarithmic function $(b > 0, b \neq 1)$ and let $u > 0$ and $w > 0$ be real numbers.

- **Product Rule:** $g(u+w) = g(u)g(w)$. In other words, $\log_b(uw) = \log_b(u) + \log_b(w)$
- **Quotient Rule:** $g\left(\dfrac{u}{w}\right) = g(u) - g(w)$. In other words, $\log_b\left(\dfrac{u}{w}\right) = \log_b(u) - \log_b(w)$
- **Power Rule:** $g(u^w) = wg(u)$. In other words, $\log_b(u^w) = w\log_b(u)$

In English 6.7. (Algebraic Properties of Exponents) For any base b and real numbers $x > 0$ and $y > 0$:

- $\log_b(xy) = \log_b(x) + \log_b(y)$
- $\log_b\left(\dfrac{x}{y}\right) = \log_b(x) - \log_b(y)$
- $\log_b(x^y) = y\log_b(x)$

Problem Solving Tip 2. Memorize the rules from theorem 6.7

1. **Worked Example**: Expand the logarithmic expression: $\ln(x^3y^2)$

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2. Expand the logarithmic expression: $\log_2\left(\dfrac{128}{x^2+4}\right)$

3. Expand the logarithmic expression: $\log(1.23 \times 10^{37})$

4. Expand the logarithmic expression: $\ln\left(\dfrac{\sqrt{z}}{xy}\right)$

5. Expand the logarithmic expression: $\log_5(x^2 - 25)$

6. Expand the logarithmic expression: $\log(1000x^3y^5)$

7. Expand the logarithmic expression: $\log_3\left(\dfrac{x^2}{81y^4}\right)$

8. Expand the logarithmic expression: $\ln\left(\sqrt[4]{\dfrac{xy}{ez}}\right)$

9. Simplify the logarithmic expression: $4\ln(x) + 2\ln(y)$

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10. Simplify the logarithmic expression: $\log_2(x) + \log_2(y) - \log_2(z)$

11. Simplify the logarithmic expression: $\log_3(x) - 2\log_3(y)$

12. Simplify the logarithmic expression: $2\ln(x) - 3\ln(y) - 4\ln(z)$

13. Simplify the logarithmic expression: $\log_5(x) - 3$

14. Simplify the logarithmic expression: $3 - \log(x)$

15. Simplify the logarithmic expression: $\log_7(x) + \log_7(x-3) - 2$

16. Simplify the logarithmic expression: $\ln(x) + \frac{1}{2}$

Textbook Theorem 6.8. (Change of Base Formulas) Let $a, b > 0$, $a, b \neq 1$.

- $a^x = b^{x \log_b(a)}$ for all real numbers x.
- $\log_a(x) = \dfrac{\log_b(x)}{\log_b(a)}$ for all real numbers $x > 0$.

Textbook Theorem 6.9. Conversion to the Natural Base: Suppose $b > 0$, $b \neq 1$. Then

- $b^x = e^{x \ln(b)}$ for all real numbers x.
- $\log_b(x) = \dfrac{\ln(x)}{\ln(b)}$ for all real numbers $x > 0$.

17. Convert 7^{x-1} to base e.

18. Convert $\log(x^2 + 1)$ to base e.

19. **Challenge Problem:** Write the expression as a single logarithm: $\log_2(x) + \log_4(x)$

Scan the QR code for a worked solution