MATH1300
Selected Challenge Problems

Volume III

SOLUTIONS

Precalculus Peer Assisted Learning

November 19, 2024

Solution Preface:

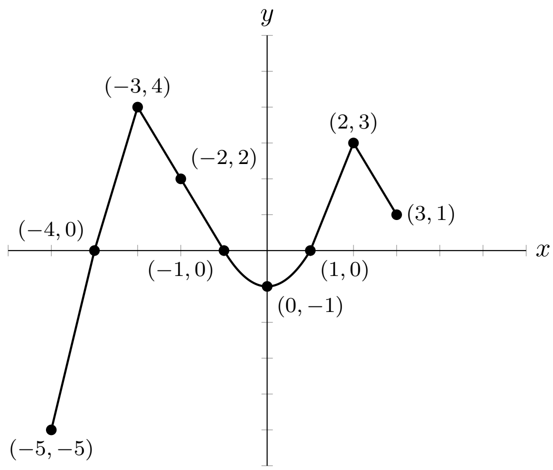
I make no guarantee that these solutions are free of typos or errors. I have done my best to ensure they are all correct; however, you should always double-check my solutions with another resource if you are unsure whether they are accurate.

Function graphs in this packet are generated by a computer and may not look exactly like the ones you draw. Remember, you are not a computer. The most important skill to develop is your ability to analyze key information about the graph and sketch a relatively accurate picture.

Roman

5.1

A Given the graph provided, answer all of the following questions.



$y = f(x)$

(a) Find the domain of f
$[-5, 3]$

(b) Find the range of f
$[-5, 4]$

(c) Find the maximum, if it exists.
$f(-3) = 4$

(d) Find the minimum, if it exists.
$f(-5) = -5$

(e) List the local maximums, if any exist.
$\{(-3, 4), (2, 3)\}$

(f) List the local minimums, if any exist.
$(0, -1)$

(g) List the intervals where f is increasing.
$[-5, -3] \cup [0, 2]$

(h) List the intervals where f is decreasing.
$[-3, 0] \cup [2, 3]$

(i) Determine $f(-2)$.
$f(-2) = 2$

(j) Solve $f(x) = 4$.
$x = -3$

(k) List the x-intercepts, if any exist.
$\{(-4, 0), (-1, 0), (1, 0)\}$

(l) List the y-intercepts, if any exist.
$(0, -1)$

(m) Find the zeros of f.
$\{-4, -1, 1\}$

(n) Solve $f(x) \geq 0$.
$[-4, -1] \cup [1, 3]$

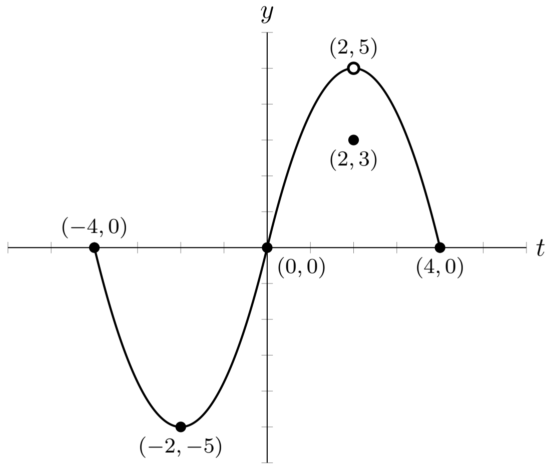
(o) Find the number of solutions to $f(x) = 1$.
4 solutions

(p) Find the number of solutions to $|f(x)| = 1$.
6 solutions

(q) Solve $(x^2 - x - 2)f(x) = 0$.
$x = \{-4, -1, 1, 2\}$

(r) Solve $(x^2 - x - 2)f(x) > 0$.
$(-4, -1) \cup (-1, 1) \cup (2, 3)$

B Given the graph provided, answer all of the following questions.



$y = g(t)$

(a) Find the domain of g.
$[-4, 4]$

(b) Find the range of g.
$[-5, 5)$

(c) Find the maximum, if it exists.
none

(d) Find the minimum, if it exists.
$g(-2) = -5$

(e) List of the local maximums, if any exist.
none

(f) List the local minimums, if any exist.
$\{(-2, -5), (2, 3)\}$

(g) List the intervals where g is increasing.
$[-2, 2)$

(h) List the intervals where g is decreasing.
$[-4, -2] \cup (2, 4]$

(i) Determine $g(2)$.
$g(2) = 3$

(j) Solve $g(t) = -5$
$t = -2$

(k) List the t-intercepts, if any exists.
$\{(-4, 0), (0, 0), (4, 0)\}$

(l) List the y-intercepts, if any exist.
$(0, 0)$

(m) Find the zeros of g.
$\{-4, 0, 4\}$

(n) Solve $g(t) \leq 0$.
$[-4, 0] \cup \{4\}$

(o) Find the domain of $G(t) = \dfrac{g(t)}{t+2}$
$[-4, -2) \cup (-2, 4]$

(p) Solve $\dfrac{g(t)}{t+2} \leq 0$
$\{-4\} \cup (-2, 0] \cup \{4\}$

(q) How many solutions are there to $[g(t)]^2 = 9$?
5 solutions

(r) Does g appear to be even, odd, or neither?
neither

5.2

A Let $f(x) = 2x$ and $g(t) = \frac{1}{2t+1}$. Compute the indicated value if it exists.

i. $(f + g)(2)$
$\frac{21}{5}$

ii. $\left(\frac{f}{g}\right)(0)$
0

iii. $(fg)\left(\frac{1}{2}\right)$
$\frac{1}{2}$

B Let f be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let g be the function defined by

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

Compute the indicated value if it exists.

i. $(g + f)(1)$
0

ii. $\left(\frac{f}{g}\right)(-2)$
does not exist

iii. $(gf)(-3)$
-8

C Let $f(x) = x - 1$ and $g(x) = \frac{1}{x-1}$, simplify the following expressions.

i. $(f + g)(x)$
$\frac{x^2-2x+2}{x-1}$

ii. $(f - g)(x)$
$\frac{x^2-2x}{x-1}$

iii. $(fg)(x)$
1

iv. $\left(\frac{f}{g}\right)(x)$
$x^2 - 2x + 1$

D Let $r(x) = \dfrac{3-x}{x+1}$.

i. Find nontrivial[1] functions f and g so that $r = fg$.
Multiple solutions possible, one example: $f(x) = 3 - x$ and $g(x) = \frac{1}{x+1}$.

E Let $f(x) = -3x + 5$.

i. Find and simplify the difference quotient using the formula: $\frac{f(x+h)-f(x)}{h}$
-3

F Let $f(x) = x - x^2$.

i. Find and simplify the difference quotient using the formula: $\frac{f(x+h)-f(x)}{h}$
$-2x - h + 1$

[1]Functions like $f(x) = 1$ do not count.

5.3

A Let $f(x) = 4x + 5$ and $g(t) = \sqrt{t}$, compute the following compositions, if any exist.

i. $(g \circ f)(0)$
$\sqrt{5}$

ii. $(f \circ f)(2)$
57

iii. $(g \circ f)(-3)$
non real answer

B Let f be the function defined by

$$f = \{(-3, 4), (-2, 2), (-1, 0), (0, 1), (1, 3), (2, 4), (3, -1)\}$$

and let g be the function defined by

$$g = \{(-3, -2), (-2, 0), (-1, -4), (0, 0), (1, -3), (2, 1), (3, 2)\}$$

Compute the indicated value if it exists.

i. $(f \circ g)(3)$
4

ii. $(f \circ g)(-3)$
2

iii. $g(f(g(0)))$
-3

iv. $f(f(f(f(f(1)))))$
3

C Let $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$. Simplify the indicated composition.

i. $(g \circ f)(x)$
$3x^2 - 3x - 2$

ii. $(f \circ g)(t)$
$9t^2 - 33t + 31$

D Let $f(x) = x^2 - x - 1$ and $g(t) = \sqrt{t - 5}$. Simplify the indicated composition.

i. $(g \circ f)(x)$
$\sqrt{x^2 - x - 6}$

ii. $(f \circ g)(t)$
$t - 6 - \sqrt{t - 5}$

E Let $f(x) = -2x$, $g(t) = \sqrt{t}$, and $h(s) = |s|$. Simplify the indicated composition.

i. $(f \circ g \circ h)(s)$
$-2\sqrt{|s|}$

ii. $(h \circ f \circ g)(t)$
$2\sqrt{t}$

iii. $(g \circ h \circ f)(x)$
$\sqrt{2|x|}$

F Write $c(x) = \dfrac{x^2}{x^4 + 1}$ as a composition of two or more non-identity functions.
Let $f(x) = x^2$ and $g(x) = \frac{x}{x^2+1}$, then define $w(x) = (g \circ f)(x)$.

5.4

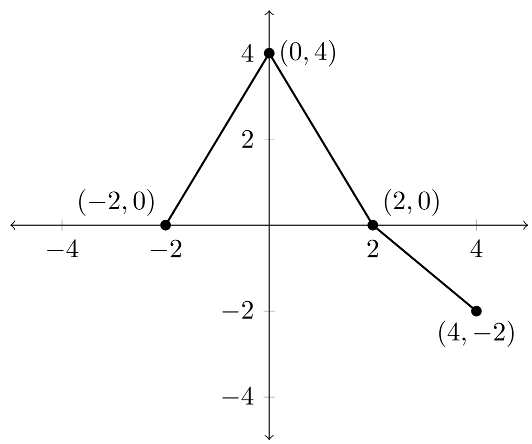
A Suppose $(2,-3)$ is on the graph of $y=f(x)$. Using function transformations, find a point on the graph of $y=3f(2x)-1$.
$(1,-10)$

B Suppose $(2,-3)$ is on the graph of $y=f(x)$. Using function transformations, find a point on the graph of $y=5f(2x+1)+3$.
$\left(\frac{1}{2},-12\right)$

C Suppose $(2,-3)$ is on the graph of $y=f(x)$. Using function transformations, find a point on the graph of $f\left(\dfrac{7-2x}{4}\right)$.
$\left(-\frac{1}{2},-3\right)$

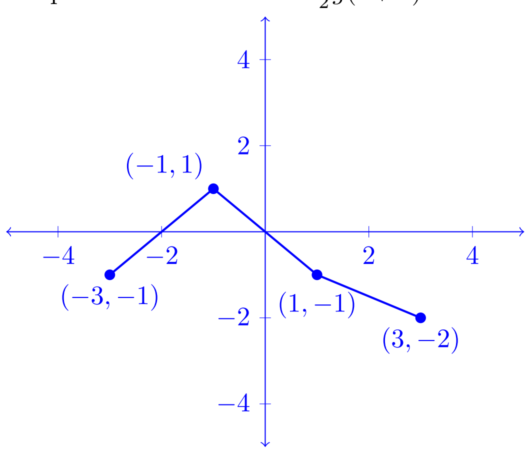
D Suppose $(2,-3)$ is on the graph of $y=f(x)$. Using function transformations, find a point on the graph of $\dfrac{4-f(3x-1)}{7}$.
$(1,1)$

E Given the graph $y = g(t)$

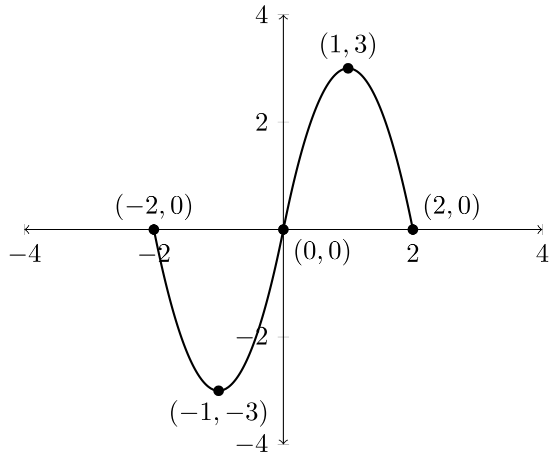


$y = g(t)$

i. Graph the transformation $\frac{1}{2}g(t+1) - 1$

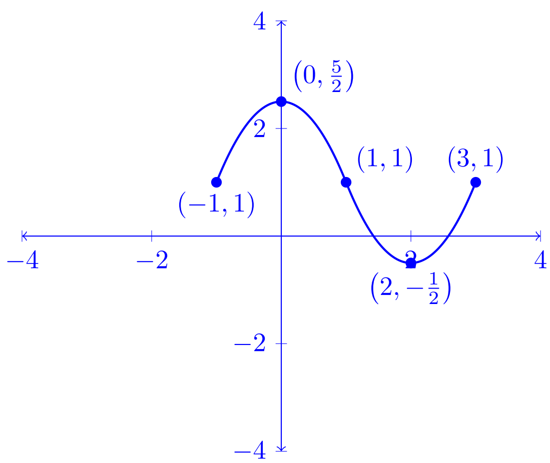


F Given the graph $y = S(t)$



$y = S(t)$

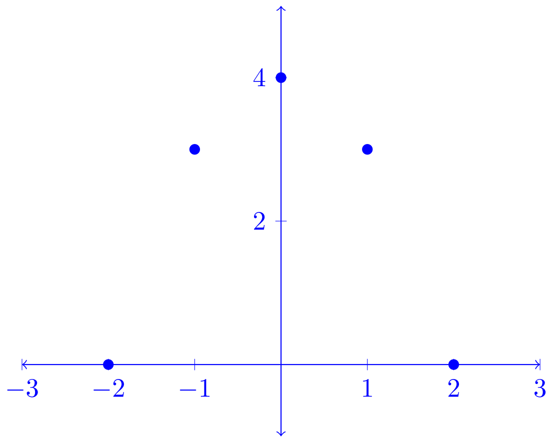
i. Graph the transformation $y = \frac{1}{2}S(-t+1)+1$



5.5

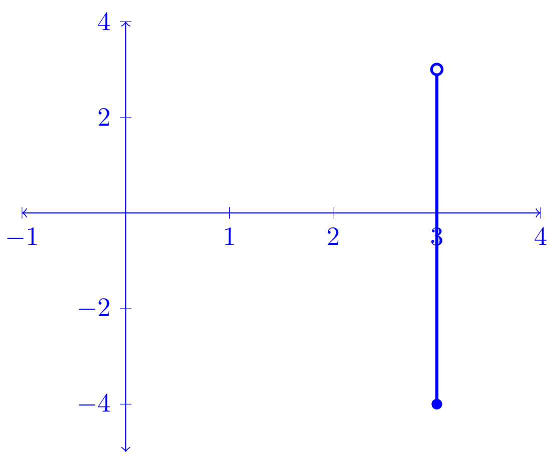
A Graph the indicated relation in the xy-plane.

i. $\{(n, 4 - n^2) \mid n = 0, \pm 1, \pm 2\}$



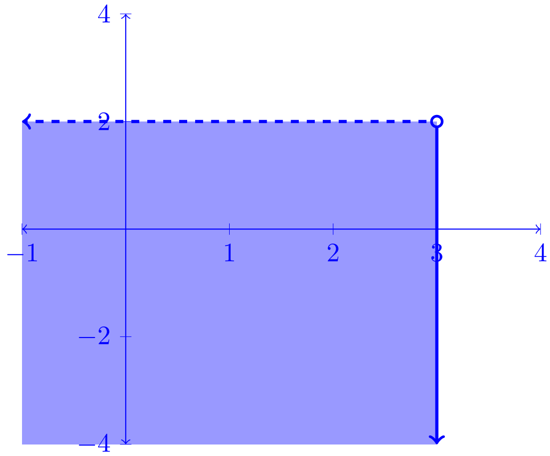
B Graph the indicated relation in the xy-plane.

i. $\{(3, y) \mid -4 \leq y < 3\}$

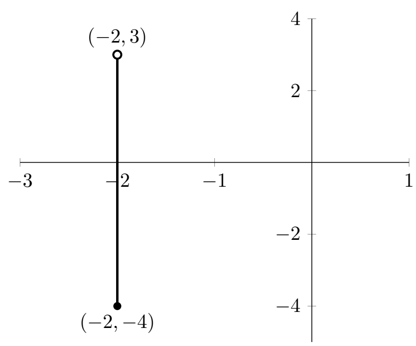


C Graph the indicated relation in the xy-plane.

i. $\{(x, y) \mid x \le 3, y < 2\}$

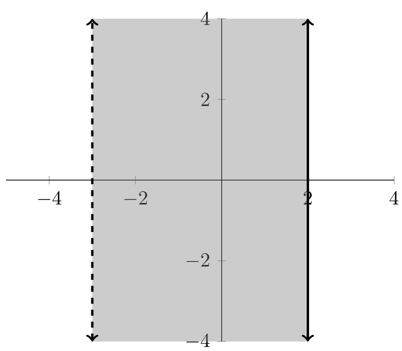


D Describe the given relation using set-builder notation.



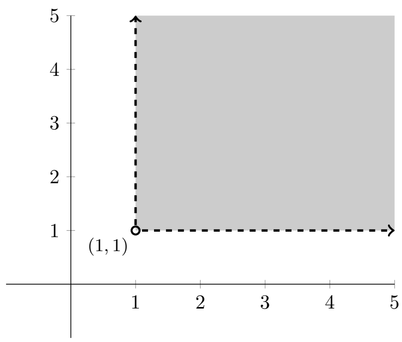
$\{(-2, y) \mid -4 \le y < 3\}$

E Describe the given relation using set-builder notation.



$\{(x, y) \mid -3 < x \le 2\}$

F Describe the given relation using set-builder notation.

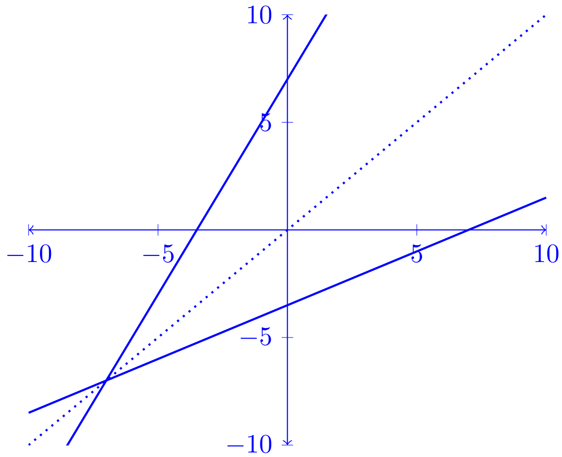


$\{(x, y) \mid 1 < x, 1 < y\}$

5.6

A Let $f(x) = 2x + 7$ and $g(x) = \dfrac{x-7}{2}$.

i. Graph $f(x)$ and $g(x)$ on a coordinate plane.



ii. Are $f(x)$ and $g(x)$ inverse? Justify your answer.
Yes, $f(x)$ and $g(x)$ are reflected over the line $y = x$

B Let $g(t) = \dfrac{t-2}{3} + 4$.

i. Show that $g(t)$ is one-to-one.

ii. Find the inverse of $g(t)$.
$g^{-1}(t) = 3t - 10$

C Let $f(x) = \sqrt{3x-1} + 5$.

i. Show that $f(x)$ is one-to-one.

ii. Find the inverse of $f(t)$.
$f^{-1}(x) = \frac{1}{3}(x-5)^2 + \frac{1}{3}, x \geq 5$

D Let $f(x) = \sqrt[5]{3x-1}$

i. Show that $f(x)$ is one-to-one.

ii. Find $f^{-1}(x)$.
$f^{-1}(x) = \frac{1}{3}x^5 + \frac{1}{3}$

E Let $h(x) = \dfrac{2x-1}{3x+4}$

i. Show that $h(x)$ is one-to-one

ii. Find $h^{-1}(x)$.

$f^{-1}(x) = \dfrac{4x+1}{2-3x}$

F* Under what conditions is $f(x) = mx + b$, $m \neq 0$ its own inverse? Prove your answer.

Proof.

Let $f(x) = mx + b$ with $m \neq 0$. Then $f^{-1}(x) = \frac{x-b}{m} = \frac{1}{m}x - \frac{b}{m}$. For f to be its own inverse we need to verify that $f(x) = f^{-1}(x)$ or in other words that $mx + b = \frac{1}{m}x - \frac{b}{m}$. This yeilds two equations which must both be true.

$$m = \frac{1}{m} \tag{1}$$

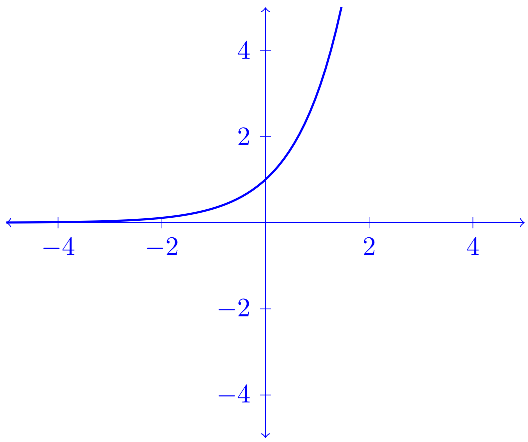
$$b = -\frac{b}{m} \tag{2}$$

Solving for (1) we obtain $m^2 - 1 = 0$ which yields $m = \pm 1$ and solving for (2) we obtain $m = -1$. However the number of solutions for m depends on the value of b. If $b \neq 0$ then m must be -1, however if $b = 0$ either $m = 1$ or $m = -1$ will work. □

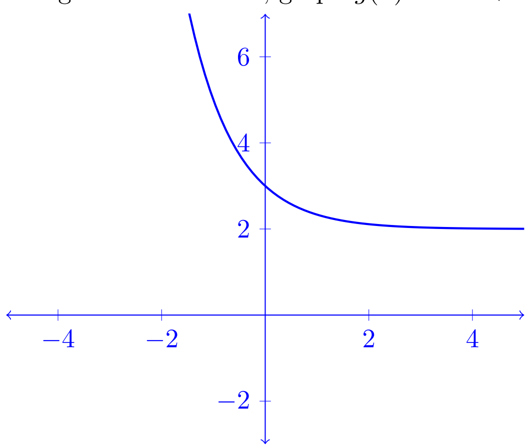
6.1

A Let $f(x) = 3^x$.

i. Sketch the graph of $f(x)$.

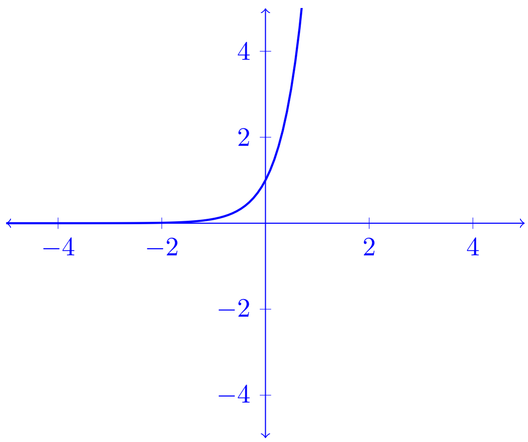


ii. Using transformations, graph $g(x) = 3^{-x} + 2$.

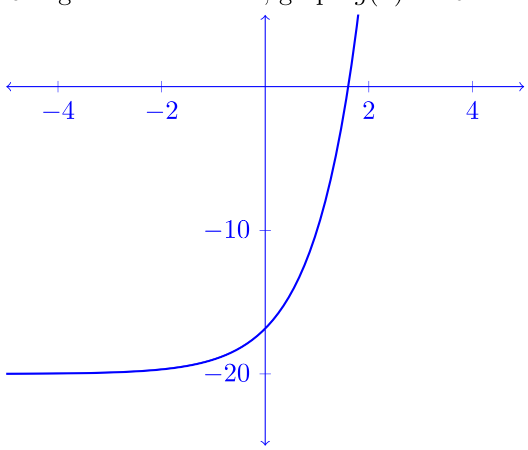


B Let $f(x) = 10^x$

i. Sketch the graph of $f(x)$.

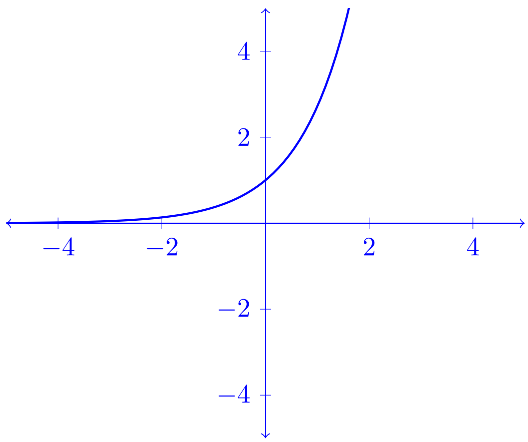


ii. Using transformations, graph $g(x) = 10^{\frac{x+1}{2}} - 20$.

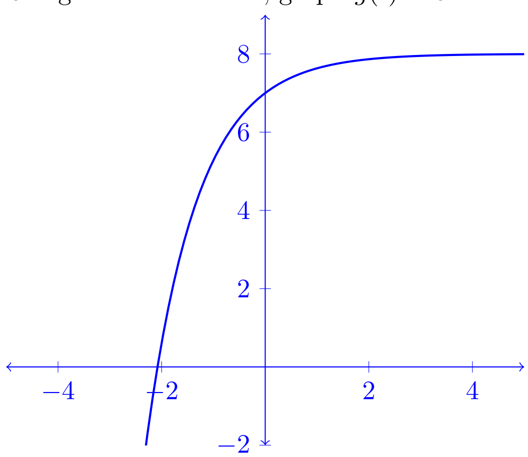


C Let $f(t) = e^t$

i. Sketch the graph of $f(t)$.



ii. Using transformations, graph $g(t) = 8 - e^{-t}$.



D State the domain of $T(x) = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

$(-\infty, \infty)$

6.2

A Rewrite the expression: $\log(100) = 2$, so that it does not contain a logarithm.
$100 = 10^2$

B Evaluate $\log_2(32)$.
5

C Evaluate $\log_4(8)$.
$\frac{3}{2}$

D Find the domain of $f(x) = \log_7(t^2 + 9t + 18)$.
$(-\infty, -6) \cup (-3, \infty)$

E Find the domain of $f(x) = \ln(x^2 + 1)$.
$(-\infty, \infty)$

F Find the domain of $g(t) = \ln(7 - t) + \ln(t - 4)$.
$(4, 7)$

6.3

A Expand and simplify: $\ln\left(\frac{\sqrt{z}}{xy}\right)$.

$\frac{1}{2}\ln(z) - \ln(x) - \ln(y)$

B Expand and simplify: $\ln\left(\sqrt[4]{\frac{xy}{ez}}\right)$.

$\frac{1}{4}\ln(x) + \frac{1}{4}\ln(y) - \frac{1}{4} - \frac{1}{4}\ln(z)$

C Write $\frac{1}{2}\log_3(x) - 2\log_3(y) - \log_3(z)$ as a single logarithm.

$\log_3\left(\frac{\sqrt{x}}{y^2 z}\right)$

D Write $\log_5(x) - 3$ as a single logarithm.

$\log_5\left(\frac{x}{125}\right)$

E Write $\log_2(x) + \log_4(x)$ as a single logarithm.

$\log_2(x^{3/2})$

F* With the product rule given, prove the quotient rule and power rule for logarithms.

Proof.

Power Rule: $\log_b(x^y) = \log_b(\underbrace{x \times \cdots \times x}_{y \text{ times}}) = \underbrace{\log_b(x) + \cdots + \log_b(x)}_{y \text{ times}} = y \times \log_b(x)$

Quotient Rule: $\log_b\left(\frac{x}{y}\right) = \log_b\left(x\frac{1}{y}\right) = \log_b\left(xy^{-1}\right) = \log_b(x) + \log_b(y^{-1})$
$= \log_b(x) + (-1 \times \log_b(y)) = \log_b(x) - \log_b(y)$

□

6.4

A Solve $2^{(t^3-t)} = 1$.
$t = \{-1, 0, 1\}$

B Solve $3^{7x} = 81^{4-2x}$.
$x = \frac{16}{15}$

C Solve $e^{2t} = e^t + 6$.
$t = \ln(3)$

D* Solve $7^{3+7x} = 3^{4-2x}$.
$x = \frac{4\ln(3)-3\ln(7)}{7\ln(7)+2\ln(3)}$

E Solve $e^{-x} - xe^{-x} \geq 0$, write your answer in interval notation.
$(-\infty, 1]$

F Solve $(1 - e^t)t^{-1} \leq 0$, write your answer in interval notation.
$(-\infty, 0) \cup (0, \infty)$

6.5

A Solve $10\log\left(\frac{x}{10^{-12}}\right) = 150$.
10^3

B Solve $3\ln(t) - 2 = 1 - \ln(t)$.
$t = e^{3/4}$

C Solve $\ln(x+1) - \ln(x) = 3$.
$x = \frac{1}{e^3-1}$

D Solve $\ln(t^2) = (\ln(t))^2$.
$t = \{1, e^2\}$

E Solve $\frac{1-\ln(t)}{t^2} < 0$, write your answer in interval notation.
(e, ∞)

F* Solve $\ln(t^2) \leq (\ln(t))^2$, write your answer in interval notation.
$(0, 1] \cup [e^2, \infty)$