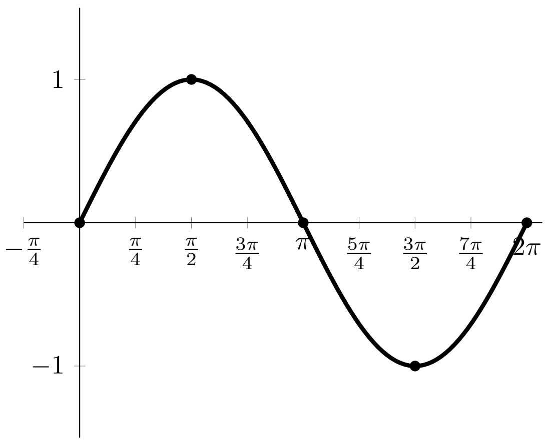
7.3 | Graphs of Sine and Cosine

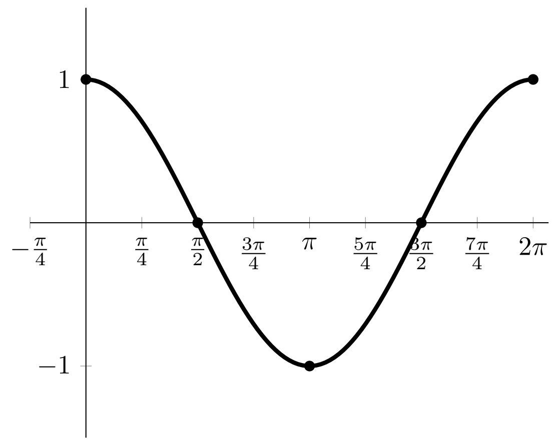
Review of Sine and Cosine: Before we get into what the graphs of sine and cosine look like. Let's briefly review what the functions are doing. First, both functions take a *radian angle as an input.* Then, the cosine function will tell you the corresponding x coordinate of the input angle on the unit circle, and the sine function tells you the corresponding y coordinate of the input angle on the unit circle.

Textbook Theorem 7.4. Domain and Range of the Cosine and Sine Functions:

- The function $f(t) = \sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
- The function $g(t) = \cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$



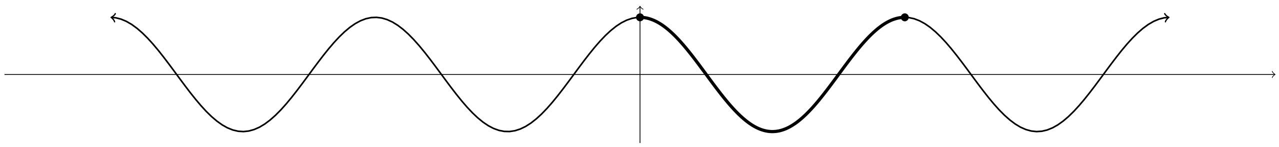
$f(t) = \sin(t)$



$g(t) = \cos(t)$

Understanding Sine and Cosine Graphs: Let's think again for a moment about how we might think about these graphs. Recall that the input to either of these functions is a radian angle measure. This is why the x-axis is commonly shown with tick marks in radians. So as we travel along the x-axis in either case, we are increasing (or decreasing if we move left) the amount of radians we are inputting into our function. Each function will then give us back either an x or y value on the unit circle as the y-axis output. This gives us the wave pattern. We can think of this wave as a sort of "tracing" of a circle, but stretched out along a graph.

Periodic Functions: One very important feature of both sine and cosine graphs is that the function are periodic. The specific mathematical definition for a periodic function is that given a function f, there is a value p for which $f(x+p) = f(x)$ for all values of x. We then call p the period of f. Think about how this works with sine and cosine: once our radian input makes one full rotation around the circle, we are now dealing with angles which are coterminal to smaller ones. Graphically, this will be the same output as a smaller radian input. Therefore, the sine and cosine functions are both periodic, and thus their graphs repeat with the same wave pattern forever.



The graph of $g(t) = \cos(t)$.

Fundamental Cycle: Because we know that sine and cosine functions repeat with period 2π forever, when we graph sine and cosine functions, we often just graph one cycle which we call the fundamental cycle. This is the function from 0 through 2π, with axis intercepts labeled, and peaks and valleys labeled. Both sine and cosine will have 5 unique points labeled, but on the fundamental cycle sine has three axis intercepts and two peaks and valleys, and cosine has two axis intercepts and three peaks and valleys.

Textbook Theorem 7.5. Properties of the Cosine and Sine Functions:

- The function $f(t) = sin(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is odd
  - has period 2π
- The function $g(t) = cos(t)$
  - has domain $(-\infty, \infty)$
  - has range $[-1, 1]$
  - is continuous and smooth
  - is even
  - has period 2π
- Conversion formulas: $\sin\left(t + \frac{\pi}{2}\right) = \cos(t)$ and $\cos\left(t - \frac{\pi}{2}\right) = \sin(t)$

We now introduce a couple of important terms used to describe sine and cosine functions. Starting with one we are already familiar with.

Period: The period of a sine or cosine function is the value (in radians) in which the function makes one full cycle around the unit circle.

Phase Shift: The phase shift can be thought of a horizontal shift. The reason we refer to it as phase shift is to differentiate between a typical functions horizontal shift. If we shift a sine or cosine function by a certain amount, due to periodicity we actually might not be able to tell that anything has happened!

Amplitude: The amplitude of a sine or cosine function is the height of the curve. One important thing to keep in mind is that we measure amplitude from the center of the function (by default the x-axis), not from top to bottom.

Baseline: Baseline is a fancy word for vertical shift. It measures where the center of the function lies and serves as the starting point for measuring amplitude.

Now we introduce a theorem which will look similar to the function transformation theorem from chapter 5. This time however, we use Greek letters to describe our transformations, because trig functions are fancy, and deserve fancy letters.

Textbook Theorem 7.6. For $\omega > 0$, the graphs of

$$S(t) = A\sin(\omega t + \phi) + B \text{ and } C(t) = A\cos(\omega t + \phi) + B$$

- have period $T = \dfrac{2\pi}{\omega}$
- have amplitude $|A|$
- have phase shift $-\dfrac{\phi}{\omega}$
- have vertical shift of 'baseline' B

Applying Theorem 7.6. to Graphing: With the information we have in theorem, we can get an idea of what the fundamental cycle of our function looks like.

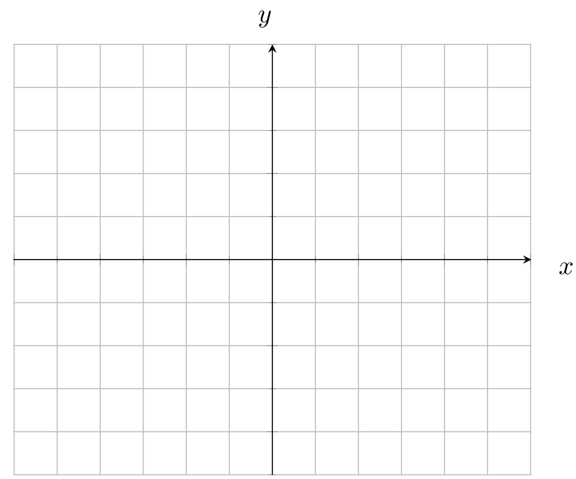
- The fundamental cycle starts at the x value $-\frac{\phi}{\omega}$ provided by the phase shift.
- The fundamental cycle ends $\frac{2\pi}{\omega}$ units away from where it started. This is one period of the function.
- The axis intercepts of the function will be centered at $y = B$ given by the baseline.
- The peaks and valleys of the function will be $|A|$ units away from B.

Problem Solving Tip 1. A clever trick for graphing:
If you look back to the pictures of $\sin(t)$ and $\cos(t)$ on page 1 of this worksheet, you may notice a pattern among the "points of interest" on both graphs of sine and cosine. Between the first point of interest and the last point of interest, the middle point of interest is exactly halfway between those two, and the remaining two points of interest are halfway between the middle point and the end points. This leads us to the following method of constructing graphs:

Use Theorem 7.6 to determine the starting point and ending point of the fundamental cycle for either sine or cosine. Then divide the graph into four equal parts. The remaining points of interest lie exactly at the 1/4 mark, the 1/2 mark, and the 3/4 mark.

Keep in mind that fundamental cycle of $\sin(t)$ starts at $(0, 0)$ and ends at $(2\pi, 0)$ while the fundamental cycle of $\cos(t)$ starts at $(0, 1)$ and ends at $(2\pi, 1)$.

1. **Worked Example:** Graph the function $f(t) = \sin\left(2t + \frac{\pi}{2}\right)$



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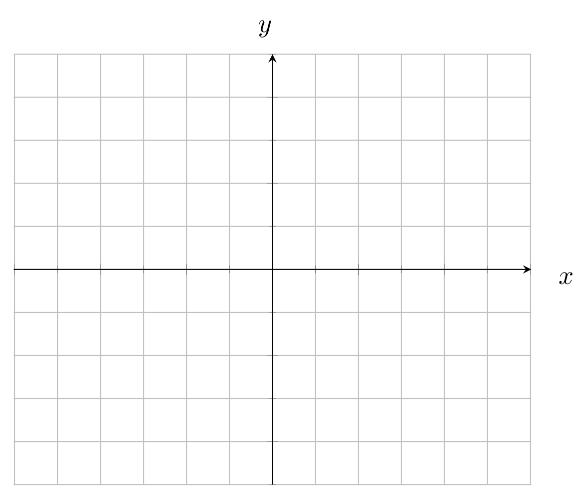
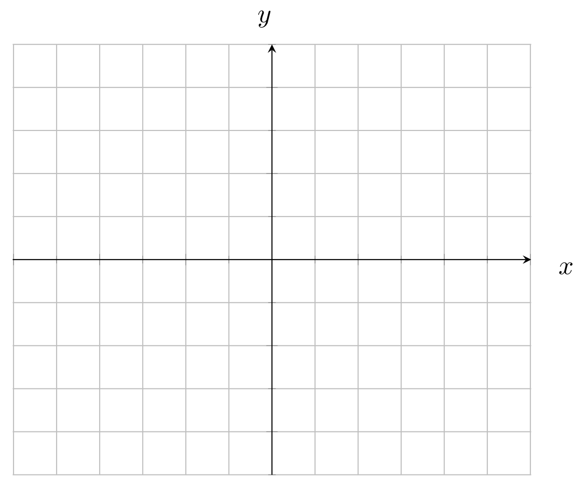
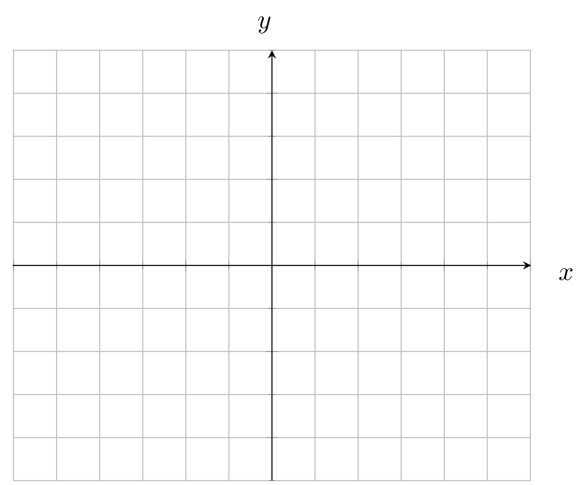
2. Graph the function $f(t) = \sin(3t)$



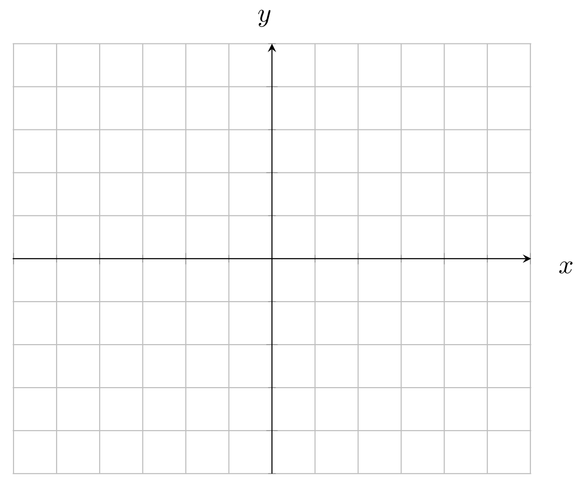
3. Graph the function $f(t) = \cos\left(t - \frac{\pi}{2}\right)$



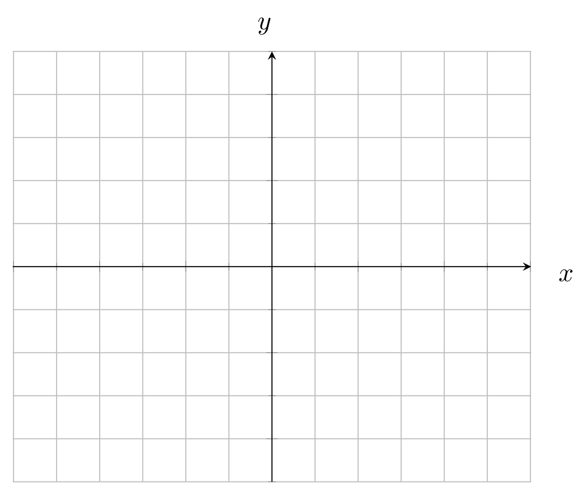
4. Graph the function $f(t) = -\sin\left(t + \frac{\pi}{3}\right)$



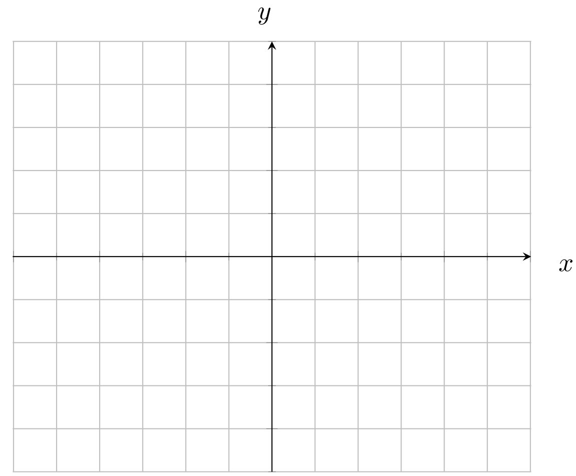
5. Graph the function $f(t) = \sin(2t - \pi)$



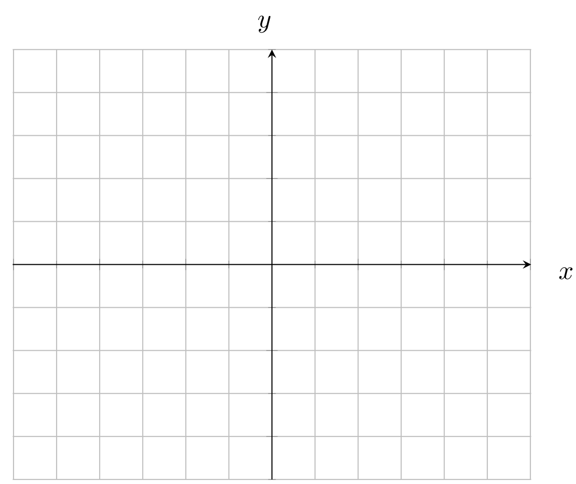
6. Graph the function $f(t) = \cos(3t - 2\pi) + 4$



7. Graph the function $f(t) = \frac{2}{3}\cos\left(\frac{\pi}{2} - 4\pi\right) + 1$



8. **Challenge Problem:** Graph the function $f(t) = |4\sin(t)|$



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