7.4 | The Circular Functions: Tangent, Secant, Cosecant, and Cotangent

In section 7.2, we saw how the functions sine and cosine take a radian angle input, and output the corresponding y or x coordinate on the unit circle respectively. Now we introduce four more functions that still take radian angle inputs, but give alternative outputs according to the following definitions:

The Circular Functions: Given a angle input θ with an intersection point on the unit circle at (x, y):

- The **sine** of θ, denoted $\sin(\theta)$, is defined by $\sin(\theta) = y$.
- The **cosine** of θ, denoted $\cos(\theta)$, is defined by $\cos(\theta) = x$.
- The **tangent** of θ, denoted $\tan(\theta)$, is defined by $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- The **secant** of θ, denoted $\sec(\theta)$, is defined by $\sec(\theta) = \dfrac{1}{x}$, provided $x \neq 0$.
- The **cosecant** of θ, denoted $\csc(\theta)$, is defined by $\csc(\theta) = \dfrac{1}{y}$, provided by $y \neq 0$.
- The **cotangent** of θ, denoted by $\cot(\theta)$, is defined by $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

Textbook Theorem 7.7. Reciprocal and Quotient Identities:

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\sec(\theta)$ is undefined.
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\csc(\theta)$ is undefined.
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$, provided $\cos(\theta) \neq 0$; if $\cos(\theta) = 0$, $\tan(\theta)$ is undefined.
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$, provided $\sin(\theta) \neq 0$; if $\sin(\theta) = 0$, $\cot(\theta)$ is undefined.

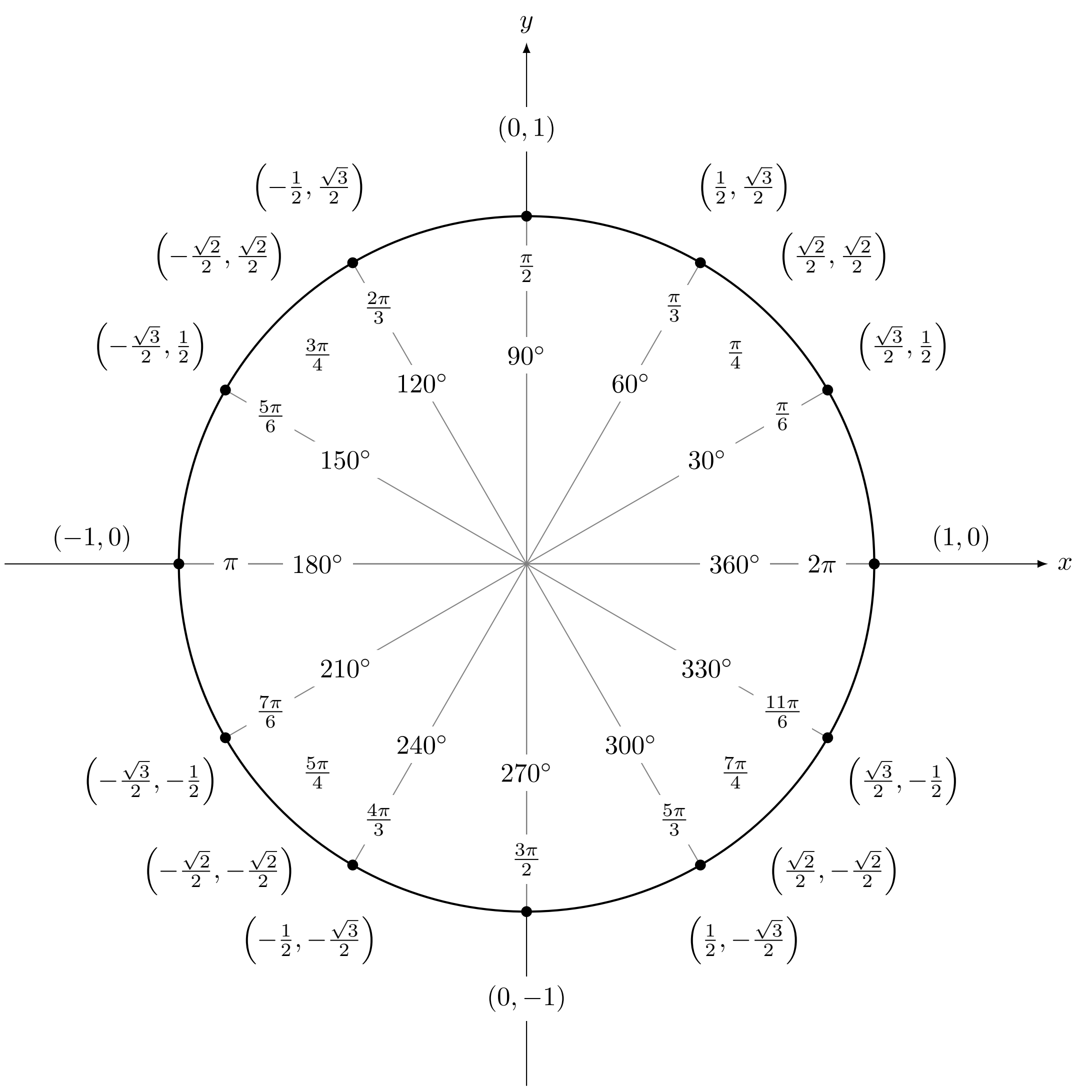
Evaluating Values of Tangent and Cotangent: Finding values of secant and cosecant is easy, just take the reciprocal of the respective function. Tangent and cotangent are more unique, and while you can compute them individually when needed, some may find it valuable to memorize a few key values:

Common Values of Tangent and Cotangent

θ (deg)	θ (rad)	$\tan(\theta)$	$\cot(\theta)$
$0°$	0	0	undefined
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$
$45°$	$\frac{\pi}{4}$	1	1
$60°$	$\frac{\pi}{3}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$
$90°$	$\frac{\pi}{2}$	undefined	0

A copy of the unit circle has also been included for easy reference.

The Unit Circle



1. Find the value of $\tan\left(\frac{\pi}{4}\right)$

2. Find the value of $\sec\left(\frac{\pi}{6}\right)$

3. Find the value of $\csc\left(\frac{5\pi}{6}\right)$

4. Find the value of $\cot\left(\frac{4\pi}{3}\right)$

5. Find the value of $\csc\left(\frac{\pi}{2}\right)$

6. Find the value of $\tan(117\pi)$

Textbook Theorem 7.8. Generalized Reference Angle Theorem. The values of the circular functions of an angle, if they exist, are the same, up to a sign, of the corresponding circular functions of its reference angle.
More specifically, if α is the reference angle for θ, then:

$$\sin(\theta) = \pm\sin(\alpha), \cos(\theta) = \pm\cos(\alpha), \tan(\theta) = \pm\tan(\alpha)$$
and
$$\sec(\theta) = \pm\sec(\alpha), \csc(\theta) = \pm\csc(\alpha), \cot(\theta) = \pm\cot(\alpha)$$

where the choice of $(\pm)$ depends on the quadrant in which the terminal side of θ lies.

7. Find all of the angles which satisfy the equation: $\tan(\theta) = \sqrt{3}$

8. Find all of the angles which satisfy the equation: $\sec(\theta) = 2$

9. Find all of the angles which satisfy the equation: $\csc(\theta) = -1$

10. Find all of the angles which satisfy the equation: $\cot(\theta) = -1$

Just like we were able to use a set of formulas to find angles for points not on the unit circle, we have analogous formulas for the new circular functions stated as follows:

Textbook Theorem 7.9. Suppose $Q(x, y)$ is the point on the terminal side of an angle θ (plotted in standard position) which lies on the circle of radius r, $x^2 + y^2 = r^2$. Then:

- $\sin(\theta) = \dfrac{y}{r} = \dfrac{y}{\sqrt{x^2 + y^2}}$
- $\cos(\theta) = \dfrac{x}{r} = \dfrac{x}{\sqrt{x^2 + y^2}}$
- $\tan(\theta) = \dfrac{y}{x}$, provided $x \neq 0$.
- $\sec(\theta) = \dfrac{r}{x} = \dfrac{\sqrt{x^2 + y^2}}{x}$, provided $x \neq 0$.
- $\csc(\theta) = \dfrac{r}{y} = \dfrac{\sqrt{x^2 + y^2}}{y}$, provided $y \neq 0$.
- $\cot(\theta) = \dfrac{x}{y}$, provided $y \neq 0$.

In English 7.9. Given a point $Q(x, y)$ on the angle θ, the following relationships can be used to find the values of the circular functions.

- $x^2 + y^2 = r^2$
- $r = \sqrt{x^2 + y^2}$

and

- $\sin(\theta) = \dfrac{y}{r}$
- $\tan(\theta) = \dfrac{y}{x}$
- $\csc(\theta) = \dfrac{r}{y}$
- $\cos(\theta) = \dfrac{x}{r}$
- $\sec(\theta) = \dfrac{r}{x}$
- $\cot(\theta) = \dfrac{x}{y}$

Note that functions may be undefined if the denominator of the fraction is 0.

11. **Worked Example:** Given $\tan(\theta) = \dfrac{12}{5}$ with θ in Quadrant III, find the values of all other circular functions.

Scan the QR code for a video solution

12. Given $\sin(\theta) = \frac{3}{5}$ with θ in Quadrant II, find the values of all other circular functions.

13. Given $\csc(\theta) = \frac{25}{24}$ with θ in Quadrant I, find the values of all other circular functions.

14. Given $\sec(\theta) = 7$ with θ in Quadrant IV, find the values of all other circular functions.

15. Given $\tan(\theta) = -2$ with θ in Quadrant IV, find the values of all other circular functions.

16. Given $\csc(\theta) = 5$ with $\dfrac{\pi}{2} < \theta < \pi$, find the values of all other circular functions.

17. Given $\tan(\theta) = \sqrt{10}$ with $\pi < \theta < \dfrac{3\pi}{2}$, find the values of all other circular functions.