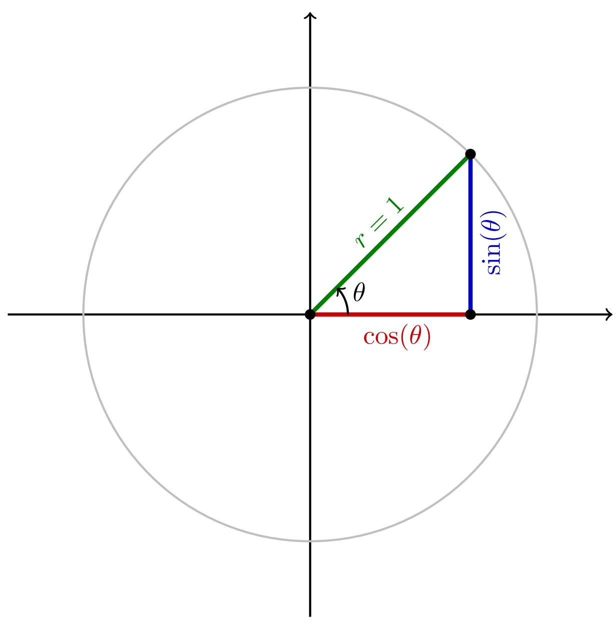
8.1 | The Pythagorean Identities

Identities: An identity, is an equation which remains true no matter the value of the variables involved. In the content of trigonometric functions, we are primarily concerned with the value of θ as an angle input. You might already be familiar with ways in which circular functions are related to each other, but now we will formally go through and present all of these identities.

Textbook Theorem 8.1. Reciprocal and Quotient Identities: The following relationships hold for all angles of θ provided each side of the equation is defined.

- $\sec(\theta) = \dfrac{1}{\cos(\theta)}$
- $\cos(\theta) = \dfrac{1}{\sec(\theta)}$
- $\csc(\theta) = \dfrac{1}{\sin(\theta)}$
- $\sin(\theta) = \dfrac{1}{\csc(\theta)}$
- $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$
- $\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$
- $\cot(\theta) = \dfrac{1}{\tan(\theta)}$
- $\tan(\theta) = \dfrac{1}{\cot(\theta)}$

Some identities are constructed via properties of the functions themselves. A easy example to understand is the following. Recall that cosine and sine functions take an input angle theta, and output the corresponding x and y values on the *unit circle* respectively. We know the unit circle has radius 1. If we treat the output of cosine (x) as one leg of a triangle, and the output of sine (y) as another leg of a triangle, we form a right angle between the two. Then, we can trace a path from the center of the unit circle to the corresponding (x, y) pair made from the values of sine and cosine. This generates a *right angle* and thus the Pythagorean theorem applies: $x^2 + y^2 = 1$. Because this holds for *all* values of x and y, we can form the following, and arguably the most famous trigonometric identity.



Textbook Theorem 8.2. The Pythagorean Identity: For any angle θ, $\cos^2(\theta) + \sin^2(\theta) = 1$.

Note About Notation: In trigonometry, when we write $\sin^2(\theta)$ we mean $(\sin(\theta))^2$, as in we are squaring the output of sine. This convention is perhaps not the most obvious, but is used widely in mathematics texts.

Now, we can take our list of reciprocal identities from theorem 8.1, and combine it with theorem 8.2, to generate a more comprehensive list of Pythagorean identities.

Textbook Theorem 8.3. The Pythagorean Identities:

1. $\cos^2(\theta) + \sin^2(\theta) = 1$.
   Common Alternate Forms:
   - $1 - \sin^2(\theta) = \cos^2(\theta)$
   - $1 - \cos^2(\theta) = \sin^2(\theta)$
2. $1 + \tan^2(\theta) = \sec^2(\theta)$, provided $\cos(\theta) \neq 0$.
   Common Alternate Forms:
   - $\sec^2(\theta) - \tan^2(\theta) = 1$
   - $\sec^2(\theta) - 1 = \tan^2(\theta)$
3. $1 + \cot^2(\theta) = \csc^2(\theta)$, provided $\sin(\theta) \neq 0$.
   Common Alternate Forms:
   - $\csc^2(\theta) - \cot^2(\theta) = 1$
   - $\csc^2(\theta) - 1 = \cot^2(\theta)$

Applying Identities: Recall in chapter 7 how if we were given the value of one trig function, we could find values of all others. We now have a new way of solving these types of problems. We can plug values of a known trig function into theorems 8.1 and 8.3 to find the values of the others. For the following practice problems, try and avoid using techniques outlined in chapter 7, instead relying on the theorems introduced in this section.

1. **Worked Example:** Given $\sin(\theta) = \frac{3}{5}$ with θ in Quadrant II, find the values of all other circular functions.

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2. Given $\tan(\theta) = \dfrac{12}{5}$ with θ in Quadrant III, find the values of all other circular functions.

3. Given $\csc(\theta) = \dfrac{25}{24}$ with θ in Quadrant I, find the values of all other circular functions.

4. Given $\sec(\theta) = 7$ with θ in Quadrant IV, find the values of all other circular functions.

Verifying Identities: If we are presented with an equation which is potentially an identity, how do we go about verifying that it does indeed hold for all values of θ? The general idea behind this, is to continuously apply known identities and/or manipulate the equation until we get the equation into the form of either an obviously true equation (like $1 = 1$), or an identity we already know. If the proposed identity is equivalent one of these two cases, then it does indeed hold for all values of θ.

Before we jump into practice problems, we will cover one more list of identities which will help equip us with extra tools that we can use to verify new identities. You might know how we can use a *conjugate* to simplify an annoying fraction. A list of conjugates is included below to help make your calculations faster, should you need to use them.

Pythagorean Conjugates:

- $(1 - \cos(\theta))(1 + \cos(\theta)) = 1 - \cos^2(\theta) = \sin^2(\theta)$
- $(1 - \sin(\theta))(1 + \sin(\theta)) = 1 - \sin^2(\theta) = \cos^2(\theta)$
- $(\sec(\theta) - 1)(\sec(\theta) + 1) = \sec^2(\theta) - 1 = \tan^2(\theta)$
- $(\sec(\theta) - \tan(\theta))(\sec(\theta) + \tan(\theta)) = \sec^2(\theta) - \tan^2(\theta) = 1$
- $(\csc(\theta) - 1)(\csc(\theta) + 1) = \csc^2(\theta) - 1 = \cot^2(\theta)$
- $(\csc(\theta) - \cot(\theta))(\csc(\theta) + \cot(\theta)) = \csc^2(\theta) - \cot^2(\theta) = 1$

Finally, let's briefly cover some tips for verifying identities, but remember, practice makes perfect!

Problem Solving Tip 1. Verifying Identities:

- Use Theorem 8.1 to substitute values of trig functions for ones which are more useful in the context of the equation.
- Use Theorem 8.3 to rearrange equations involving a squared trig function to obtain a more useful trig function to work with.
- Use the Pythagorean conjugates to simplify complicated fractions.
- Focus on one side of the equation, can you make it look like the other? If one side is giving you trouble, try working from the other. (Although sometimes working from both sides is also effective!)
- If you feel stuck, just try substituting values where you can. Sometimes making an equation consist mainly of the same trig function can lead to a better insight of what needs to be done.

For the following identities, assume all necessary quantities are defined.

5. **Worked Example:** Verify: $\dfrac{\sin(\theta)}{\cos^2(\theta)} = \sec(\theta)\tan(\theta)$

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6. Verify: $\cos(\theta)\sec(\theta) = 1$

7. Verify: $\tan(t)\cot(t) = 1$

8. Verify: $\dfrac{\cos(\theta)}{\sin^2(\theta)} = \csc(\theta)\cot(\theta)$

9. Verify: $\dfrac{\sin(x)}{1-\cos^2(x)} = \csc(x)$

10. Verify: $\dfrac{\tan(x)}{\sec^2(x)-1} = \cot(x)$

11. Verify: $9 - \cos^2(t) - \sin^2(t) = 8$

12. Verify: $\cos^2(x)\tan^3(x) = \tan(x) - \sin(x)\cos(x)$

13. Verify: $\dfrac{\cos(\theta) + 1}{\cos(\theta) - 1} = \dfrac{1 + \sec(\theta)}{1 - \sec(\theta)}$

14. Verify: $\dfrac{1-\tan(t)}{1+\tan(t)} = \dfrac{\cos(t)-\sin(t)}{\cos(t)+\sin(t)}$

15. Verify: $\tan(\theta) + \cot(\theta) = \sec(\theta)\csc(\theta)$

16. Verify: $\sin(t)(\tan(t) + \cot(t)) = \sec(t)$

17. Verify: $\dfrac{1}{1-\cos(\theta)} + \dfrac{1}{1+\cos(\theta)} = 2\csc^2(\theta)$

18. Verify: $\dfrac{\cos(\theta)}{1-\tan(\theta)} + \dfrac{\sin(\theta)}{1-\cot(\theta)} = \sin(\theta) + \cos(\theta)$

19. Verify: $\dfrac{1}{\sec(x)-\tan(x)} = \sec(x) + \tan(x)$

20. Verify: $\dfrac{1}{1+\cos(x)} = \csc^2(x) - \csc(x)\cot(x)$

21. **Challenge Problem:** Verify: $\dfrac{1-\sin(x)}{1+\sin(x)} = (\sec(x) - \tan(x))^2$

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