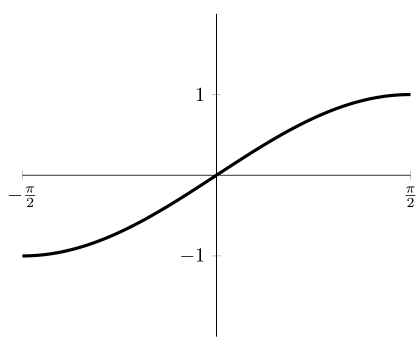
8.3 | The Inverse Circular Functions

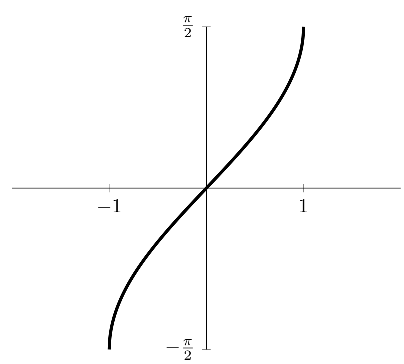
The Issue with Circular Inverses: Recall from section 5.6 that every inverse function is one-to-one, and vise versa. The issue with forming inverses for circular functions, is that none of the circular functions are one-to-one, and so we cannot form an inverse for the entire function. To fix this issue, we will restrict the domain of the circular functions to a specific domain so that the restricted function is one-to-one. This will allow us to form inverses for certain sections of circular functions.

Naming Convention: We denote the inverse circular functions by adding the phrase "arc" in front of them. So if $f(x) = \sin(x)$, then $f^{-1}(x) = \arcsin(x)$. This pattern holds for all six circular functions.

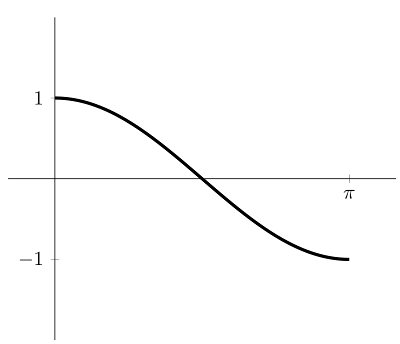
Inverse of Sine and Cosine: When we choose the restriction of sine and cosine, we will want to make the choice useful so that the inverse function can be useful for computation. This means we want to choose a section of the function that hits every y value between 1 and -1 so that every output on the unit circle can be accounted for. For sine, we will choose the restriction: $\sin(t), -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$. For cosine, we will choose the restriction: $\cos(t), 0 \leq t \leq \pi$. These functions look as follows:



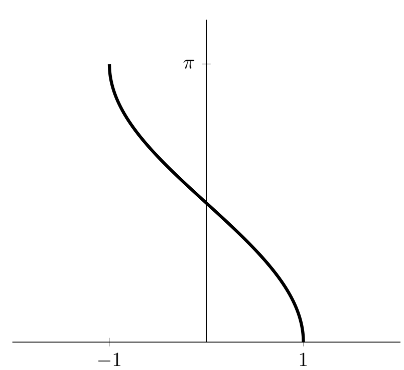
$f(t) = \sin(t), -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$



$f^{-1}(t) = \arcsin(t)$



$f(t) = \cos(t), 0 \leq t \leq \pi$



$f^{-1}(t) = \arccos(t)$

Textbook Theorem 8.12. Properties of the Arccosine and Arcsine Functions

- Properties of $F(x) = \arcsin(x)$
  - Domain: $[-1, 1]$
  - Range: $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
  - $\arcsin(x) = t$ if and only if $\sin(t) = x$ and $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $\sin(\arcsin(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arcsin(\sin(t)) = t$ provided $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$
  - $F(x) = \arcsin(x)$ is odd
- Properties of $G(x) = \arccos(x)$
  - Domain: $[-1, 1]$
  - Range: $[0, \pi]$
  - $\arccos(x) = t$ if and only if $\cos(t) = x$ and $0 \leq t \leq \pi$
  - $\cos(\arccos(x)) = x$ provided $-1 \leq x \leq 1$
  - $\arccos(\cos(t)) = t$ provided $0 \leq t \leq \pi$

Evaluating Arcsine and Arccosine: If we want to evaluate either arcsine or arccosine, it is useful to recall that two functions f and g are inverse if $f(g(x)) = x = g(f(x))$. So if we want to calculate the value of an *inverse* trig function, we can set the function equal to an unknown variable, and then we can evaluate both sides of the equation on the respective trig function. This will reduce the side of the equation with the inverse trig function to a single value, and then we can solve back for the value of a trig function using the unit circle. Just make sure that all domain restrictions are being followed within the context of the inverse function.

1. **Worked Example:** Evaluate: $\arcsin(-1)$

Scan the QR code for a worked solution

2. Evaluate: $\arcsin\left(-\dfrac{\sqrt{3}}{2}\right)$

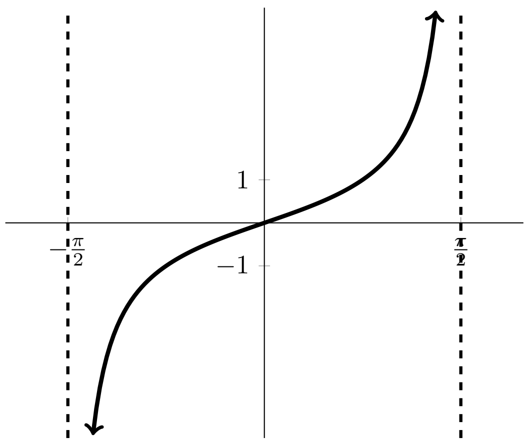
3. Evaluate: $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$

4. Evaluate: $\arccos\left(-\frac{1}{2}\right)$

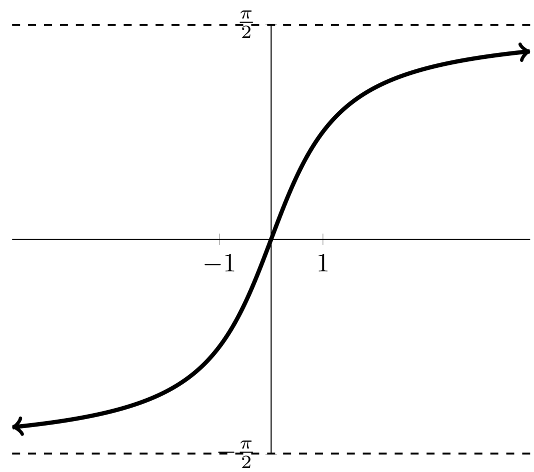
5. Evaluate: $\arccos(0)$

6. Evaluate: $\arccos\left(\frac{\sqrt{3}}{2}\right)$

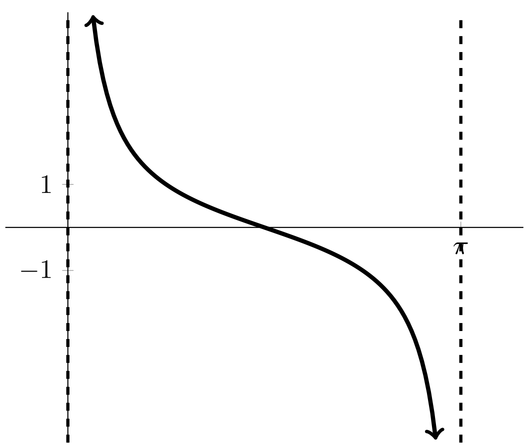
Inverse of Tangent and Cotangent: For the tangent and cotangent functions, we follow a similar technique. We restrict the domain of the function so that it is one-to-one, and then construct an inverse. Tangent is restricted similar to sine, and cotangent is restricted similar to cosine. These functions look as follows:



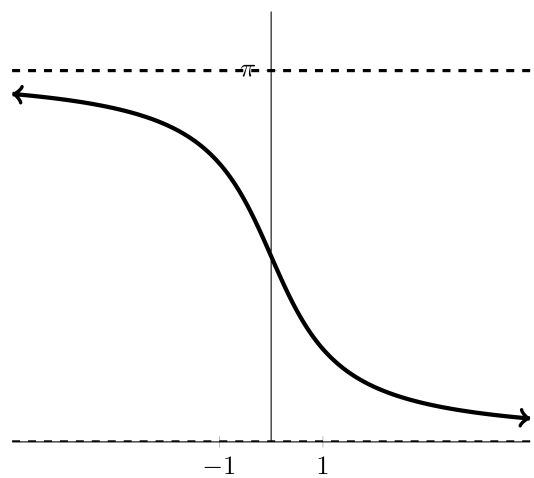
$f(t) = \tan(t), -\frac{\pi}{2} < t < \frac{\pi}{2}$



$f^{-1}(t) = \arctan(t)$



$f(t) = \cot(t), 0 < t < \pi$



$f^{-1}(t) = \text{arccot}(t)$

Textbook Theorem 8.13. Properties of the Arctangent and Arccotangent Functions

- Properties of $F(x) = \arctan(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
  - as $x \to -\infty$, $\arctan(x) \to -\frac{\pi}{2}^{+}$; as $x \to \infty$, $\arctan(x) \to \frac{\pi}{2}^{-}$
  - $\arctan(x) = t$ if and only if $\tan(t) = x$ and $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $\arctan(x) = \operatorname{arccot}\left(\frac{1}{x}\right)$ for $x > 0$
  - $\tan(\arctan(x)) = x$ for all real numbers x
  - $\arctan(\tan(t)) = t$ provided $-\frac{\pi}{2} < t < \frac{\pi}{2}$
  - $F(x) = \arctan(x)$ is odd
- Properties of $G(x) = \operatorname{arccot}(x)$
  - Domain: $(-\infty, \infty)$
  - Range: $(0, \pi)$
  - as $x \to -\infty, \operatorname{arccot}(x) \to \pi^{-}$; as $x \to \infty, \operatorname{arccot}(x) \to 0^{+}$
  - $\operatorname{arccot}(x) = t$ if and only if $\cot(t) = x$ and $0 < t < \pi$
  - $\operatorname{arccot}(x) = \arctan\left(\frac{1}{x}\right)$ for $x > 0$
  - $\cot(\operatorname{arccot}(x)) = x$ for all real numbers x
  - $\operatorname{arccot}(\cot(t)) = t$ provided $0 < t < \pi$

Evaluating Arctangent and Arccotangent: To evaluate arctangent and arccotangent we use a really similar idea to evaluating arcsine and arccosine. Simply set the value of the inverse function equal to an unknown, and then take the respective function of both sides. Cancel the function and its inverse, and then solve back for tangent or cotangent.

7. Evaluate: $\arctan\left(-\sqrt{3}\right)$

8. Evaluate: $\arctan(-1)$

9. Evaluate: $\arctan(0)$

10. Evaluate: $\operatorname{arccot}\left(-\frac{\sqrt{3}}{3}\right)$

11. Evaluate: $\operatorname{arccot}\left(\sqrt{3}\right)$

Evaluating Arcsec and Arccosec: The textbook does not provide definitions or theorems for inverse secant and cosecant functions, however we don't need them to evaluate inverse secant and cosecant functions. We can instead employ the following method.

Suppose we wish to evaluate $\text{arcsec}(\alpha)$. We start by setting this value to an unknown: $\text{arcsec}(\alpha) = x$. Now we can take the secant of both sides of the equation to obtain: $\alpha = \sec(x)$. Finally, we can use the reciprocal identity from section 8.1 to convert the expression to a question about cosine: $\frac{1}{\alpha} = \cos(x)$.

12. Evaluate: $\text{arcsec}(2)$

13. Evaluate: $\text{arcsec}(\sqrt{2})$

14. Evaluate: $\text{arccsc}(1)$

Inverse circular functions as solutions to equations: In chapter 7, we saw questions that asked us to state all solutions to a given equation. However, if one of these questions happen to have nothing to do with the unit circle, or even nothing to do with π, having inverse circular functions gives us a way to write down a solution[1].

If we are asked to write down the solution of every equation to $\sin(\theta) = \alpha$ (where $0 < \alpha < 2\pi$), we can do the following: Find the value of θ by taking the arcsine function of both sides of the equation. Now, $\theta = \arcsin(\alpha)$. However we still need to account for two things. First, θ can be more than simply $\arcsin(\alpha)$, it could also be the reference angle produced by $\pi - \arcsin(\alpha)$, so we denote these two possibilities.[2] Second, all angles which are coterminal to both of these cases, are also all valid solutions. So we can state finally that $\theta = \arcsin(\alpha) + 2\pi k$ OR $\theta = \pi - \arcsin(\alpha) + 2\pi k$ where k is an integer.

15. **Worked Example:** Write all solutions to the following equation: $\sin(\theta) = \frac{7}{11}$

Scan the QR code for a video solution

16. Write all solutions to the following equation: $\cos(\theta) = -\frac{2}{9}$

[1]Even if that solution means virtually nothing to us humans.
[2]There are two possibilities in the first place due to the fact that sine has identical solutions in both quadrant I and quadrant II.