0.3 | Completing the Square

There are more than one variation on the completing the square method. The method you may have learned in class might vary from the one demonstrated here. There is no one method that is better than another, you may use whichever method you find most helpful for you (provided that it works).

What do we use Completing the Square for?: Completing the square is a method that is used to transform a quadratic equation from general form ($ax^2 + bx + c$) into standard form ($a(x - h)^2 + k$). Standard form has a much easier to identify vertex, and is much easier to use when graphing an equation. Therefore, if we are asked to graph a quadratic that is in general form, completing the square to turn it into standard form is a great starting point.

The Steps: The step by step process to completing the square goes as follows:

1. Divide the quadratic by a
2. Add the value $\left(\frac{b}{2a}\right)^2$ after bx and subtract the value $\left(\frac{b}{2a}\right)^2$ after c.
3. Factor the first three terms $\left[x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2\right]$ into $\left(x + \frac{b}{2a}\right)^2$.
4. Simplify the remaining constants $\left[\frac{c}{a} - \left(\frac{b}{2a}\right)^2\right]$
5. Multiply everything back by a

That was so confusing: The steps to completing the square themselves are not... the most simple to understand. Let's go through an example together, and explain more concretely what is going on and why we make the choices we do.

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Setup: Before you begin, it is important to make sure all terms are on one side of the equation, and that the other side is equal to zero. If this is not the case for whatever reason, make sure to move all the terms to one side before completing the square. Let's use the following equation for our example:

$$2x^2 + 4x + 4 = 0$$

Step 1: In the first step, we will divide the entire quadratic by a. In order for completing the square to work, the term x^2 must not have any coefficient attached to it, so we can temporarily fix this by dividing everything out by a, as long as we remember to put it back in later.

$$\frac{2x^2 + 4x + 4}{2} = \frac{0}{2}$$

In our case $a = 2$, so the coefficient 2 in front of x^2 will reduce, and the two values of 4 will reduce to 2. When dividing zero by anything, it simply remains as 0.

$$x^2 + 2x + 2 = 0$$

Step 2: Before we continue with step 2, we should write down the value of $(\frac{b}{2a})^2$. However, we already divided everything out by a, so it is often simpler to take our b value from the reduced equation after step 1, and only calculate $(\frac{b}{2})^2$. In our case, $b = 2$, so:

$$\left(\frac{b}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = (1)^2 = 1$$

Once we have this value of $(\frac{b}{2})^2$, we will add and subtract it at two points in the equation. We add and subtract, so that the overall value of the equation is not changed, but writing it down will help to complete a factorization later. We add and subtract 1 in two places like so:

$$x^2 + 2x + 1 + 2 - 1 = 0$$

Step 3: Next, we take advantage of the way in which we added and subtracted $(\frac{b}{2})^2$. We start by grouping together the first three terms of the function:

$$\left[x^2 + 2x + 1\right] + 2 - 1 = 0$$

Notice that the terms in between the brackets is now a new quadratic. This new quadratic *always* factors into $(x + \frac{b}{2})^2$. You will never have to waste time factoring when completing the square! In our case we calculated $\frac{b}{2} = 1$ so:

$$(x + 1)^2 + 2 - 1 = 0$$

Step 4: Now we can turn our attention to the constants left over after factoring. We can simply combine these together as like terms.

$$(x + 1)^2 + 1 = 0$$

Step 5: Our last step will only need to be done if we had to divide by a in the beginning. If a started out as 1, we can stop at step 4, but recall that in our case we divided out a 2 from every term in order to reduce our x^2 term. Now we will adjust for that by multiplying the entire equation by a again. For us, $a = 2$ and we can do:

$$2\left[(x + 1)^2 + 1\right] = 2[0]$$

Simply distribute the 2, and note that since the other side of the equation is 0 as it remains unaffected by multiplication:

$$2(x + 1)^2 + 2 = 0$$

This completes the method of completing the square. These steps can be done on any quadratic to transfer it from general form to standard form. Try completing the square on your own with the following exercises.

1. Convert the following equation to standard form: $x^2 + 6x + 5 = 0$

2. Convert the following equation to standard form: $x^2 + 8x - 9 = 0$

3. Convert the following equation to standard form: $x^2 - 6x + 9 = 0$

4. Convert the following equation to standard form: $x^2 + 4x - 7 = 0$

5. Convert the following equation to standard form: $x^2 - 5x - 24 = 0$

6. Convert the following equation to standard form: $x^2 - 8x + 15 = 0$

7. Convert the following equation to standard form: $4x^2 - 4x + 17 = 0$

8. Convert the following equation to standard form: $9x^2 - 12x + 13 = 0$

9. Convert the following equation to standard form: $4x^2 - 4x + 5 = 0$

10. Convert the following equation to standard form: $4x^2 - 8x + 1 = 0$