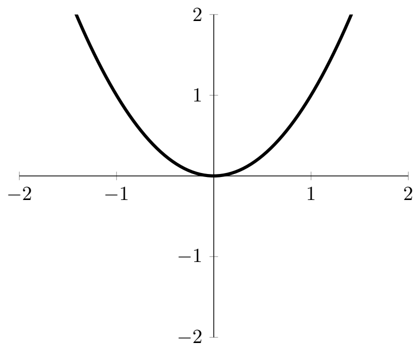
1.4 | Quadratic Functions

Quadratic Functions: A quadratic function is of the form

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$. The quadratic function takes the shape of a *parabola*. When a is positive the parabola opens upward, as pictured to the right. When a is negative, the parabola opens downward, which would appear as a reflection over the x-axis.

Vertex: The vertex of a parabola is the lowest or highest point of the curve. The vertex on the example graph provided to the right is the point $(0, 0)$.



$f(x) = x^2$

General Form: General form is what you probably are most used to seeing, it is the definition given above of

$$f(x) = ax^2 + bx + c$$

where $a \neq 0$. The advantage of general form is that it makes the three values of a, b and c readily identifiable. The main disadvantage is that it is difficult to identify the vertex from the equation alone. The vertex *can* be found by identifying the point $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$.

Standard Form: Standard form is an alternative form that looks like the following:

$$f(x) = a(x - h)^2 + k$$

where $a \neq 0$. This form has the benefit of making the vertex of the parabola easily identifiable as the point (h, k), however the values of b and c are less apparent.

Converting from General to Standard Form: To convert from general to standard form we need to use a method called *completing the square*. This method, however, would take too long to describe in full here. If you need a review on completing the square, please see my dedicated worksheet to the topic by going to `romansimkins.com/pal/worksheets`.

1. Convert the following quadratic to standard form, then identify the vertex: $f(x) = x^2 - 2x - 8$

2. Convert the following quadratic to standard form, then identify the vertex: $g(t) = 2t^2 - 4t - 1$

The Quadratic Formula: To find the zeros (the places where the function crosses the x-axis) of a quadratic function, we can use the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The advantage of this is that no matter how complicated the values of a, b and c are, we can always brute force solve for solutions.[1] The disadvantage, is that this method is often overkill for the types of equations we will deal with in this course.

3. Use the quadratic formula to solve the following quadratic: $h(s) = -3s^2 + 5s + 4$

[1] The quadratic formula can often produce very ugly looking answers, but as long as all your calculations are correct the solutions will be correct.

The zeros of a function are the places in which the function crosses the x-axis. Quadratic functions can have either 0, 1, or 2 zeros. Think of the following scenarios:

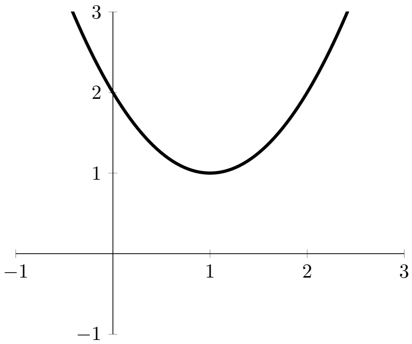


figure 1.1

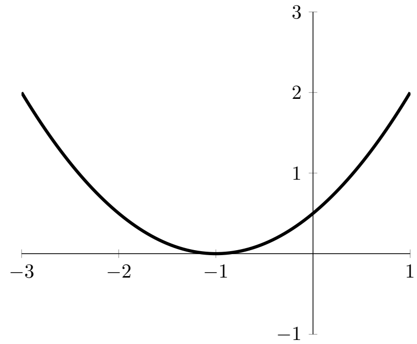


figure 1.2

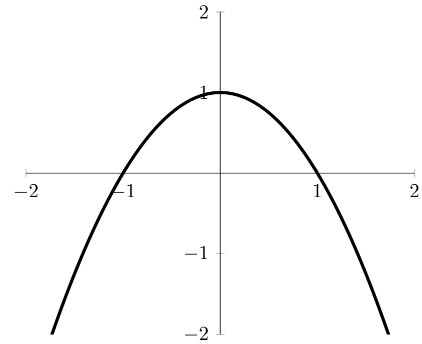


figure 1.3

In *figure 1.1* the parabola does not cross the x-axis, so if we tried to plug the values of its equation in the quadratic formula, we would run into answers that don't make any sense (more on that shortly). Take a look at *figure 1.2*, this parabola only has one point that touches the x-axis, so when solving for the zeros of this function we would get one (repeated) answer that works. Finally, *figure 1.3* has two points that touch the x-axis, and if we plug into the quadratic formula we would get two working answers.

Real Zeros: When solving for the zero of a function, we are interested in *real* answers. By this, we simply mean something that we can represent on a number line like $x = 6$ or $x = 12.345$. Sometimes, we may get solutions that cannot be represented on a number line. The cause of this is when we try and take the square root of a negative number. If we said that $x = \sqrt{-2}$, there is no way to represent -2 on a number line (think: what two real numbers can you multiply together and get -2? hint: there are none[2]). This means that $x = \sqrt{-2}$ is a non-real zero, and we disregard it as a solution to our function.

Discriminant of a Quadratic Function: We can generalize the conversation about real and non-real solutions using something called the *discriminant*. The discriminant of a quadratic function is $b^2 - 4ac$, and it allows for an algebraic method of identifying the number of real solutions to a quadratic. To find the number of real solutions we can utilize the following method.

Given a quadratic function of the form $f(x) = ax^2 + bx + c$ the following holds:

- If $b^2 - 4ac > 0$ then f has two distinct real zeros
- If $b^2 - 4ac = 0$ then f has one real zero
- If $b^2 - 4ac < 0$ then f has no real zeros

[2] Some of you may be familiar with the imaginary number i, which lets us represent values like $\sqrt{-2}$ (in this case $i\sqrt{2}$). This course *does not* cover imaginary numbers, and so we will simply regard solutions like $\sqrt{-2}$ as non-real solutions

4. Use the discriminant to identify the number of real solutions to the following quadratic: $f(x) = x^2 - 2x - 8$

5. Use the discriminant to identify the number of real solutions to the following quadratic: $g(t) = 2t^2 - 4t - 1$

Graphing Quadratic Functions: To graph a quadratic function, we first want to analyze it algebraically and answer some big questions about the function, that will then provide the context clues to drawing a complete graph. It's a good idea to keep a mental checklist of these items, and create a road map of what we will be looking for in any quadratic that we graph. These big questions are the following:

- What is the vertex?
- Is the parabola opening up, or down?
- Where are the zeros (x-intercepts)?
- Where is the y-intercept?

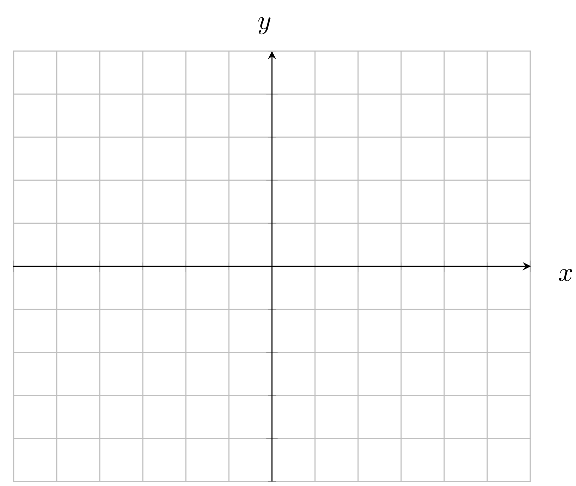
Graphing functions can be intimidating, but if you are able to answer these questions *before* graphing, you will find that it is much easier to fill in the rest of the detail. Let's look at each question and the roadmap to solving it.

- What is the vertex?
  - Recall that to find the vertex, we will use a different formula depending on what form the quadratic is in. The vertex formula for general form is $(-\frac{b}{2a}.f(-\frac{b}{2a}))$. The vertex for standard form is much simpler, being only (h, k).
- Is the parabola opening up, or down?
  - To identify which way the parabola opens, we simply need to analyze the coefficient a, which, lucky for us, is the same in both general and standard form. If a is positive, then the parabola opens upward. If a is negative, the parabola opens downward. If a is zero, we no longer have a quadratic (but a linear)!

- Where are the zeros (x-intercepts)?
  - To find zeros (also called roots), we can take advantage of the fact that every point on lying on the x-axis will have a y value of 0. So in an equation like $y = ax^2 + bx + c$, we can set $y = 0$ and solve. This works for every function, not just quadratics. So if we want to solve for zeros, we start by setting the whole equation equal to zero and solving.[3]
- Where is the y-intercept?
  - To find the y-intercept we can use the same reasoning, in that any point lying on the y-axis will have an x value of 0. So to solve the y-axis for any equation, we can set $x = 0$ and solve. Because multiplying zero reduces everything to zero, this often makes the calculation very simple.

Problem Solving Tip 1. Graphing Quadratics As you get better at working with quadratic functions, you will not need all four bits of information to draw a complete graph. You will also be introduced to other methods of graphing that you may find more useful. However if you find yourself stuck on a test question, these four bits of information will get you a head start in any graphing problem.

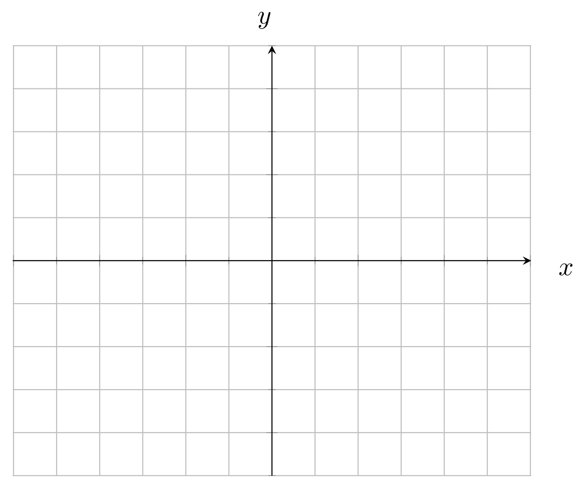
6. **Worked Example**: Sketch the graph of the function $f(x) = (x - 1)^2 - 1$



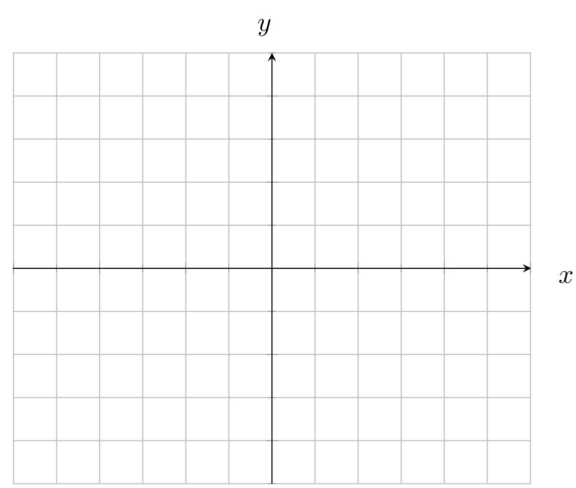
Scan the QR code for a video solution.

[3]Actually solving this equation for zero can be tricky for those who aren't knowledgeable about the methods. Recall that these methods are factoring, using the quadratic formula, or solving with a calculator (which is discouraged in this class). If you need a review on factoring, please see my worksheet dedicated to factoring on `romansimkins.com/pal/worksheets`

7. Sketch the graph of the function: $g(x) = x^2 + 2x - 3$



8. Sketch the graph of the function: $f(x) = x^2 - 2x - 8$



Solving Inequalities: To solve an inequality involving a quadratic, we can follow these steps:

- Move all terms to one side of the equation
- Factor the quadratic
- Make a *sign diagram*[4]
- Analyze the sign diagram and interpret based on the sign

9. **Worked Example**: Solve the inequality: $x^2 + 2x - 3 \geq 0$

Scan the QR code for a video solution.

10. Solve the inequality: $t^2 + 9 < 6t$

[4]If you are unfamiliar with how sign diagrams work, I have a dedicated worksheet for them on my website at `romansimkins.com/pal/worksheets`. I will use a sign diagram in the first worked example, so you are welcome to follow along there as well.

11. Solve the inequality: $u^2 + 4 \leq 4u$

12. Solve the inequality: $x > x^2$

13. Solve the inequality: $5t + 4 \leq 3t^2$