3.2 | Graphs of Rational Functions

Note: This section uses sign diagrams. If you need a complete review of sign diagrams, please see worksheet 0.5 on my website: `https://romansimkins.com/pal/worksheets/`.

In section 3.1 we looked at a series of methods that we can use to analyze important features of rational functions. The only thing we are missing from being able to graph rational functions is a way to tell where a rational function is positive or negative. This of course is where we can use sign diagrams

The process for making a sign diagram for a rational function is nearly the same as with polynomial functions, however, we have a new feature of rational functions that we didn't have before, and that is values that are excluded from the domain via a vertical asymptote. Vertical asymptotes are still locations where functions can move from positive to negative or vice versa, so we can treat them almost identically to that of zeros, but we will want to make a note for ourselves that they are missing from the domain when we interpret the sign diagram and graph the function.

But first, we should talk about something that seems trivial, but that we have not actually discussed surrounding rational functions.

Finding Zeros of Rational Functions: Given a rational function, factor the numerator and denominator and then cancel as many terms as possible. In this reduced function, set the numerator equal to zero and solve. These values will be zeros of the rational function.

Constructing a Sign Diagram for a Rational Functions:

- Find any vertical asymptotes, place these values on the sign diagram with a "?" symbol above them.
- Find any zeros, place these values on the sign diagrams with a "0" above them.
- Compute the sign diagram using any method of your choosing, you may treat asymptotes and zeros *exactly the same* in this step for the sake of computation.
- Interpret the sign diagram. Keep in mind that values marked with a "?" will always be removed from the domain.

Finally, let's talk about the general method to graphing rational functions. This step by step process can be applied to most problems, provided that a rational function has a factorable numerator and denominator and is in the form of one fraction.

Graphing Rational Functions: Let $r(x) = \dfrac{p(x)}{q(x)}$ be a rational function.

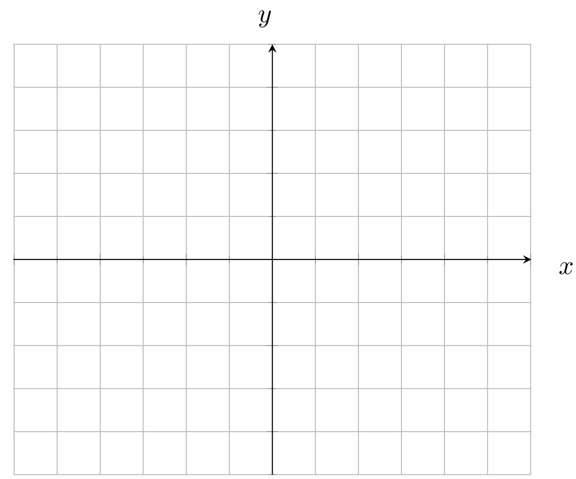
- Factor $p(x)$ and $q(x)$.
- Cancel any terms you can, any factors completely removed from the denominator will be a **hole**.
- Plug the values of any holes into the reduced form of $r(x)$ to find their corresponding y-values.
- Set the numerator equal to zero and solve, these solutions will be the **zeros** of $r(x)$.
- Set the denominator equal to zero and solve, these solutions will be the **vertical asymptotes** of $r(x)$.
- Find the **horizontal asymptote** or **slant asymptote**, if either exist.
- Set $x = 0$ and solve to find the y-intercepts.
- Construct a **sign diagram** and find the intervals on which $r(x)$ is positive or negative.
- Sketch the function based on all of the clues provided in the steps above.

Filling in the Information: Rational functions are quite unique, and while it can be straight forward to follow each step and write down the information, it can be difficult at first to understand how to use these bits of information to draw the graph of a rational function. Part of getting good at this is lots of practice and seeing examples, however I will include a few tips that can help.

- Use the asymptotes as "guide rails" for your function. If a horizontal asymptote is present, start drawing your function at the endpoints of the horizontal asymptotes.
- Pay close attention to the values on your sign diagram. The sign diagram is extremely important to being able to choose the correct sides of asymptotes to draw on.
- Keep in mind that a function will *never* touch a vertical asymptote, use this to your advantage.
- The only place that your function can cross the x-axis is at the location of a zero.

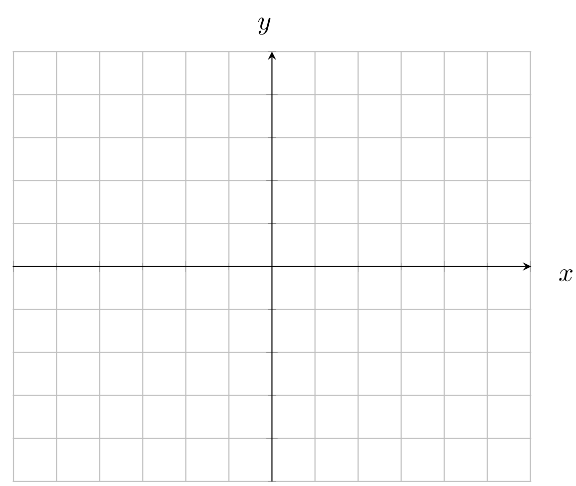
Due to the difficulty in graphing rational functions, I highly recommend that you look at the worked example present on this worksheet.

1. **Worked Example:** Sketch the function: $f(x) = \dfrac{(x-1)(x+1)}{(x+1)(x+2)}$

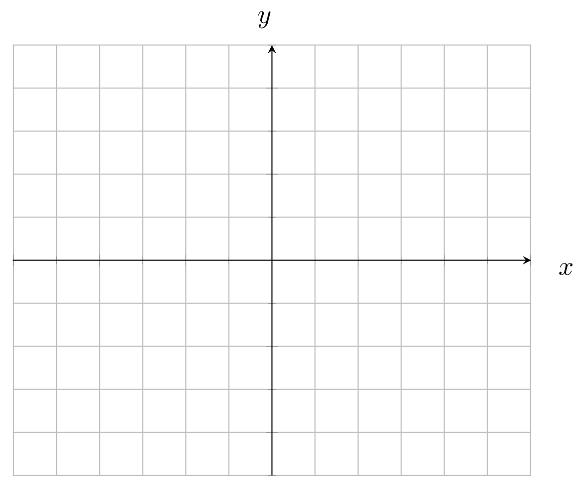


Scan the QR code for a video solution.

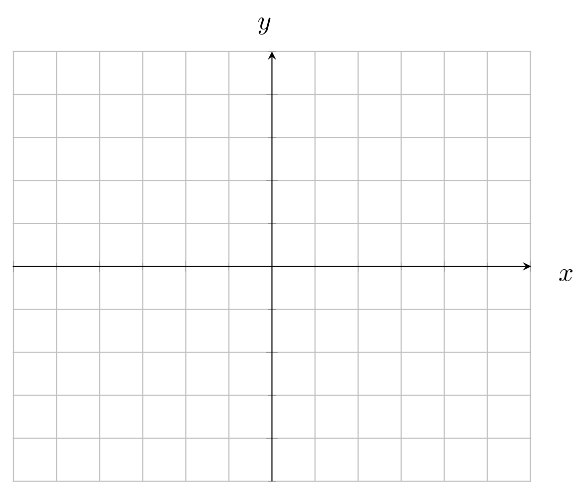
2. Sketch the function: $f(x) = \dfrac{4}{x+2}$



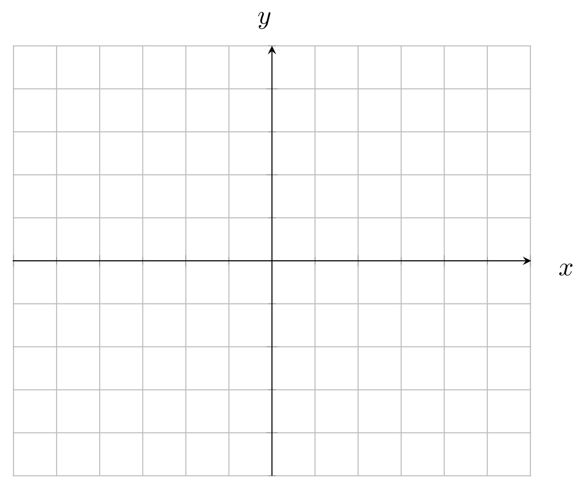
3. Sketch the function: $f(x) = \dfrac{5x}{6 - 2x}$



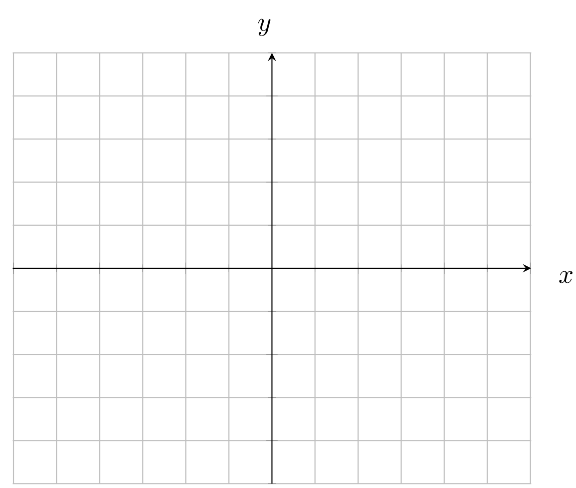
4. Sketch the function: $g(t) = \dfrac{1}{t^2 + t - 12}$



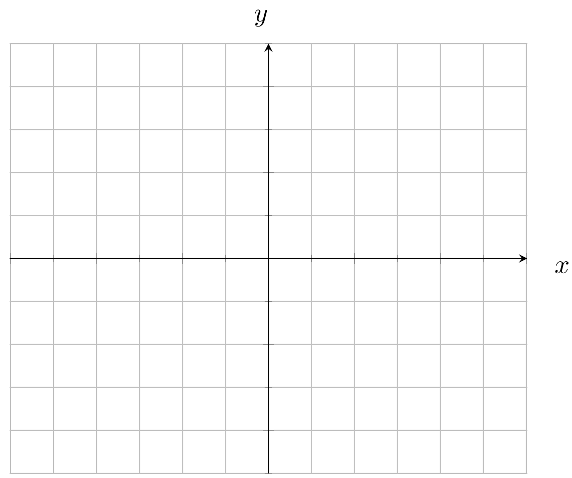
5. Sketch the function: $r(z) = \dfrac{z}{z^2 + z - 12}$



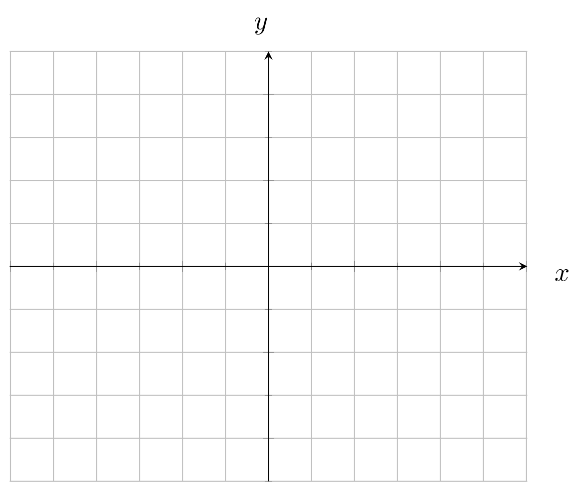
6. Sketch the function: $f(x) = \dfrac{4x}{x^2 + 4}$



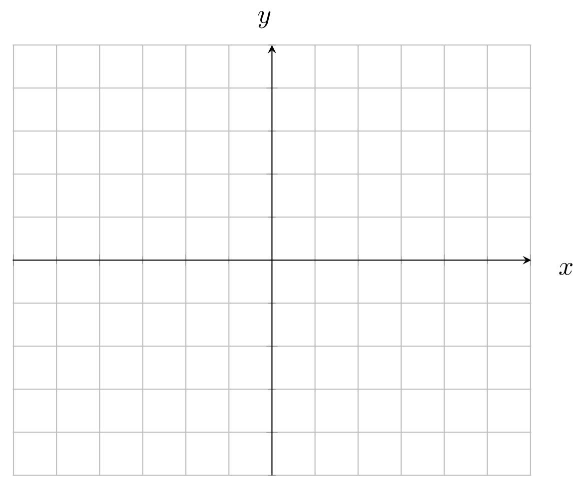
7. Sketch the function: $g(t) = \dfrac{t^2 - t - 12}{t^2 + t - 6}$



8. Sketch the function: $r(z) = \dfrac{z^2 - z - 6}{z + 1}$



9. Sketch the function: $f(x) = \dfrac{x^3 + 2x^2 + x}{x^2 - x - 2}$



10. Sketch the function: $g(t) = \dfrac{t^2 - 2t + 1}{t^3 + t^2 - 2t}$