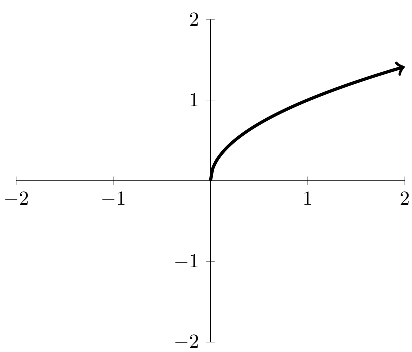
4.1 | Root and Radical Functions

Root Functions: A root function is of the form

$$f(x) = \sqrt[n]{x}$$

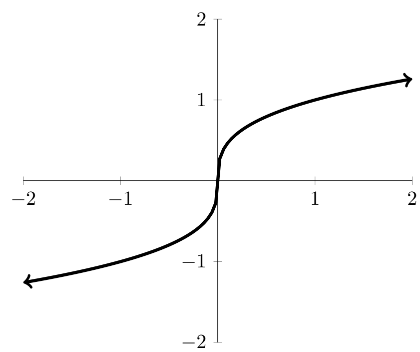
where n is a positive integer and $n \geq 2$. We can refer to root functions of the type shown above as the nth principal root function. The visual appearance and domain of the root function changes depending on n being even or odd.

Even Root Functions: Has domain $[0, \infty)$



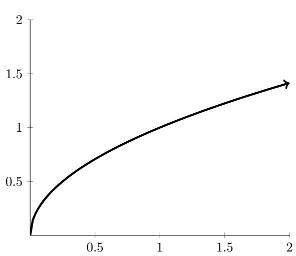
$f(x) = \sqrt[2]{x}$

Odd Root Functions: Has domain $(-\infty, \infty)$

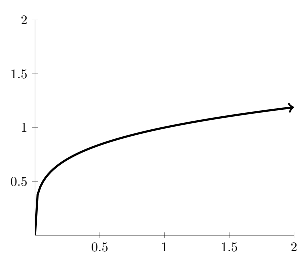


$f(x) = \sqrt[3]{x}$

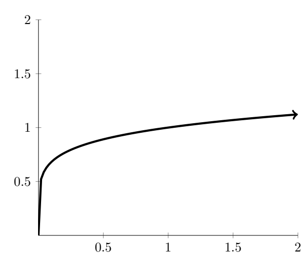
If the value of n is 2, the standard convention is to omit the value: $\sqrt[2]{n} = \sqrt{n}$ (the 2 is implied). For larger values of n, the even and odd pattern will continue, but the curve will grow sharper.



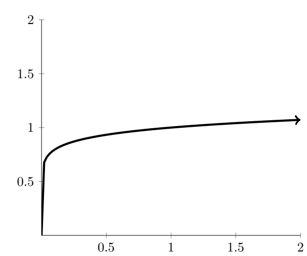
$f(x) = \sqrt{x}$



$f(x) = \sqrt[4]{x}$



$f(x) = \sqrt[6]{x}$



$f(x) = \sqrt[10]{x}$

Textbook Theorem 4.1. For real numbers a, b, h, and k with $a, b \neq 0$, the graph of $F(x) = a\sqrt[n]{bx - h} + k$ can be obtained from the graph of $f(x) = \sqrt[n]{x}$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = \sqrt[n]{x} = \sqrt[n]{x - h}$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.
   NOTE: This transforms the graph of $y = \sqrt[n]{x - h}$ to $y = \sqrt[n]{bx - h}$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = \sqrt[n]{bx - h}$ to $y = a\sqrt[n]{bx - h}$.

4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a\sqrt[n]{bx - h}$ to $y = a\sqrt[n]{bx - h} + k$.

In English 4.1. Given a function of the form $F(x) = a\sqrt[n]{bx - h} + k$, write down the parent function $P(x) = \sqrt[n]{x}$. Write down at least 3 *sample points* from the parent function, then do the following in order:

- Add h to every x-value.
- Divide all x-values by b.
- Multiply all y-values by a.
- Add k to every y-value.

Graph these new points, then trace the parent function.

Picking Sample Points: Choosing sample points on a root function is more complicated than what we have seen before. When we choose three points, we want to capture enough information to be able to trace the function properly. The catch to this is that one of our sample points will change depending on the value of n.

- **Even Root Functions**: Choose the sample points $\{(0, 0), (1, 1), (2^n, 2)\}$

This choice of $(2^n, 2)$ might seem interesting, but this point will provide you with the minimal amount of information that will help you graph the tail of the curve. For odd functions, we want to keep enough information to graph to the curve, while also providing enough information to determine the function's behavior in the other direction. We can choose between two paths here.

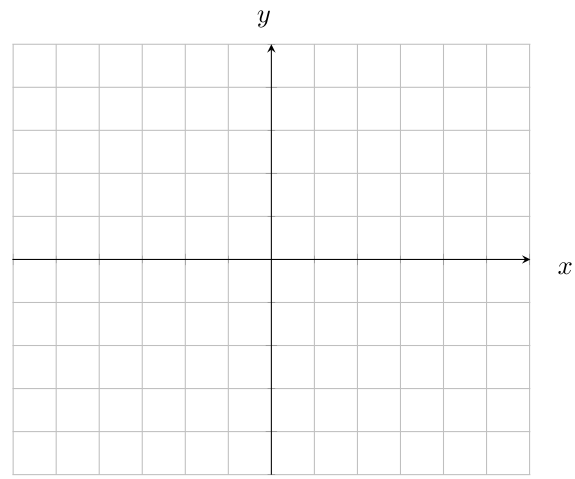
Odd Root Functions:

- Choose the sample points $\{(-2^n, -2), (-1, -1), (0, 0), (1, 1), (2^n, 2)\}$

  OR

- Choose the sample points $\{(0, 0), (1, 1), (2^n, 2)\}$ and then multiply the **result** of the points $(1, 1)$ and $(2^n, 2)$ by (-1).

1. **Worked Example:** Graph the function using transformations: $\sqrt{4-x}-1$



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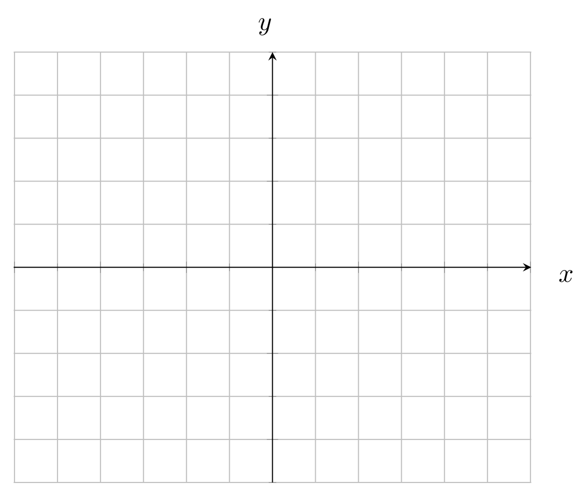
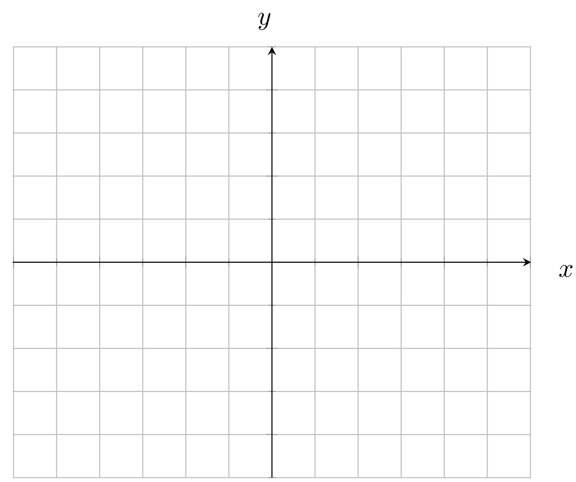
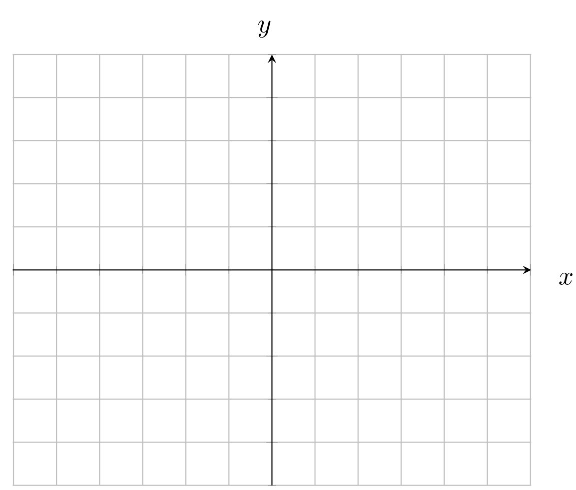
2. Graph the function using transformations: $\sqrt[3]{x-1}-2$



3. Graph the function using transformations: $-\sqrt[3]{8x+8}+4$



4. Graph the function using transformations: $-3\sqrt[4]{x-7}+1$



Textbook Theorem 4.2. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers.

Simplifying nth powers and nth roots:

- $\left(\sqrt[n]{x}\right)^n = x$
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$

Root Functions Preserve Inequality: if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

Finding Domain of Root Functions: When we go to find the domain of a root function, we need to be extra careful, especially in the case of n being even. If n is odd, the domain is automatically $(-\infty, \infty)$ and this will not be affected unless an asymptote (division by 0) is present. However, if n is even, we will need to be more careful. We want to ensure that the radical is only taking in values that are greater than or equal to 0. In other words, if we have a root function of the form $\sqrt[n]{F(x)}$ where n is even. Then the solutions to $F(x) \geq 0$ will correspond to the domain of $\sqrt[n]{F(x)}$.

Problem Solving Tip 1. To find the domain of an even root function, set everything inside the root greater than or equal to zero and solve. These solutions will be the domain of the root function (this may involve a sign diagram).

Finding the Range of Root Functions: Again, the method we use will depend on the case of n being even or odd. If n is odd, the range will be $(-\infty, \infty)$ automatically, and so we can simply state it as such. However if n is even, then we need to be careful. Even root functions have a range of $[0, \infty)$ and we can verify the lower bound of the range based on the values of k from theorem **4.1** (or the upper bound on the range if $a < 0$ as the function will reflect across the x-axis).

End Behavior: Root functions of the form $\sqrt[n]{x}$ have the following end behavior:

- **Even Root Functions:** as $x \to \infty, f(x) \to \infty$
- **Odd Root Functions:** as $x \to \infty, f(x) \to \infty$ and as $x \to -\infty, f(x) \to -\infty$

Including a Root Function in a Sign Diagram: You can also include a root function in the computation of a sign diagram. Odd root functions can be treated like a positive linear function (unless multiplied by a negative). Even root functions can also be treated like a positive linear, but you can only include the positive portion, since the root function cannot be negative. You should note the domain restriction when drawing the line for the sign diagram by starting and ending it at the domain of the even root function.

5. **Worked Example:** Find the domain of the function: $f(x) = \sqrt{1 - x^2}$

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6. Find the domain of the function: $f(x) = \sqrt{x^2 - 1}$

7. Find the domain of the function: $g(t) = t\sqrt{1 - t^2}$

8. Find the domain of the function: $f(x) = \sqrt[4]{\dfrac{16x}{x^2 - 9}}$

9. Find the domain of the function: $f(x) = \dfrac{5x}{\sqrt[3]{x^3 + 8}}$

10. Find the domain of the function: $g(t) = \sqrt{t(t+5)(t-4)}$