4.2 | Power Functions

Power Functions: A power function is of the form

$$f(x) = ax^p$$

where a and p are non zero real numbers. This may feel the exact same as function definitions we have introduced in previous sections, however, the key detail is that we are no longer restricting the value of p to an integer. This definition allows for functions like: $f(x) = x^{4/3}$, $g(t) = t^{0.4}$ or $h(w) = w^{\sqrt{2}}$.

Rational Exponents: The following formula is *very useful to know* for visualizing and analyzing power functions with rational exponents:

$$x^{m/n} = \left(\sqrt[n]{x}\right)^m = \sqrt[n]{x^m}$$

whenever $\sqrt[n]{x^m}$ is defined.

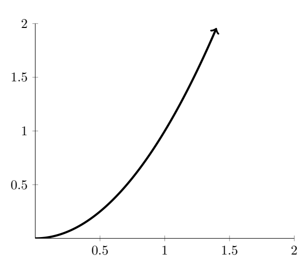
Power functions are more unique than what we have dealt with so far, but if we take some extra care, we can get a decent picture of what is going on. The first thing we need to distinguish, is even vs odd functions.

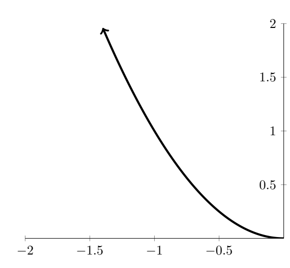
Even vs Odd Power Functions: When we look at a power function of the form $f(x) = x^{m/n}$ we want to examine n to determine if the function is even or odd. Keep in mind that we can simplify this to $f(x) = \sqrt[n]{x^m}$, and so the domain of the function will have a restriction applied, equivalent to that of a root function depending on the value of n. Just as a reminder:

- If n is even, the domain is $[0, \infty)$.
- If n is odd, the domain is $(-\infty, \infty)$.

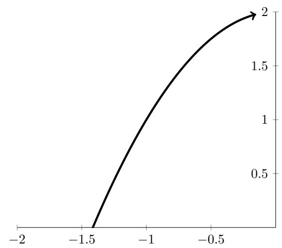
Concavity: We need more than just the information about the evenness or oddness of the power function to understand what it will look like. We use the term *concavity* to describe the behavior of a curve as it increases or decreases.

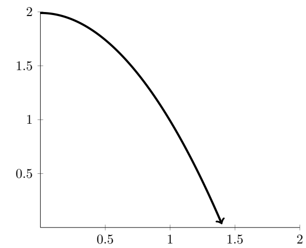
- **Concave Up**: If a function cusps upwards *regardless of if it is increasing or decreasing*, then we can describe the function as concave up. Below are two examples of concave up; one example shows an increasing function, while the other shows a decreasing function.





- **Concave Down**: If a function cusps downwards *regardless of if it is increasing or decreasing*, then we can describe the function as concave down. Below are two examples of concave down; one example shows an increasing function, while the other shows a decreasing function.





Another way to visualize this is that concave-up functions resemble a bowl that could be filled with water. You might need to rotate it slightly, but you could imagine putting items in the bowl and they would not fall out. Concave down, however, more closely resembles an upside-down bowl. Where if you tried pouring water on it, it would all fall off to the sides. The key thing to keep in mind is that concavity is independent of whether the function is increasing or decreasing; it is a separate characteristic.

Concavity of Power Functions: Finally, we can look at how to determine concavity from a power function. Before we do anything, we should ensure that the exponent for $f(x) = x^{m/n}$ has $\frac{m}{n}$ in its lowest terms (simplified). Now we want to analyze two key facts about our function.

- **Determine if $\frac{m}{n} < 1$ or $\frac{m}{n} > 1$.**
  This step is easy. If $m > n$, then $\frac{m}{n} > 1$ and if $m < n$ then $\frac{m}{n} < 1$. This will help us to determine the concavity of the function *in the first quadrant*. If $\frac{m}{n} > 1$ the concavity in the first quadrant is **concave up**. If $\frac{m}{n} < 1$ the concavity of the first quadrant is **concave down**. However we need some more information to determine concavity of the other quadrants. For this we move to our next step.

- **Determine if n is even or odd, and if m is even or odd.**
  From this point, we can look at the following cases:

  – n is even (m is even or odd)
    If n is even, then the domain of the function is restricted, and we have no other quadrants to worry about. We can simply mark down the concavity from the previous step, and we are done.

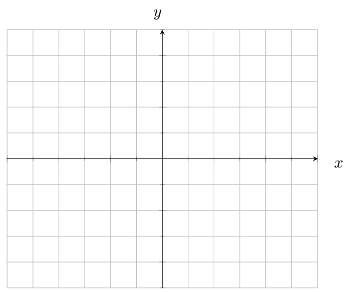
  – n is odd, m is even
    If n is odd and m is even, then the domain is defined everywhere. Record the concavity of the first quadrant from the first step, then the function will be symmetric about the y-axis and the second quadrant will *share the concavity of the first quadrant.*[1]

  – n is odd, m is odd
    If n is odd and m is odd, then the domain is again defined everywhere. Record the concavity of the first quadrant from the first step, then the function will be defined in the *third quadrant* by a reflection across both axes. The concavity in the third quadrant will be *opposite of that which was present in the first quadrant.*

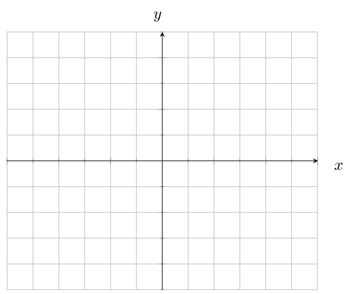
[1] The interesting case here is when n is odd, m is even, and $\frac{m}{n} < 1$. This leads to a concave down function symmetric about the y-axis (think about what this might look like). If we have the same case but $\frac{m}{n} > 1$, then we have a concave up function symmetric about the y-axis, which will look nearly identical to the even monomial functions we have dealt with in previous sections.

For the following functions, determine the concavity of the function and then sketch it on a graph. **Do not** attempt to pick points and graph the function, simply represent the concavity visually.

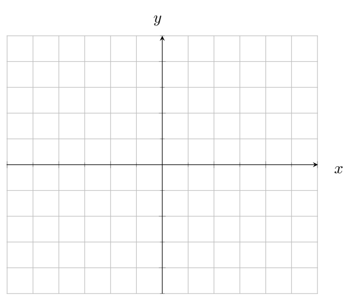
1. $f(x) = x^{\frac{2}{4}}$



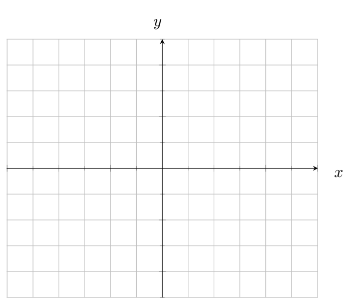
2. $f(x) = x^{\frac{2}{5}}$



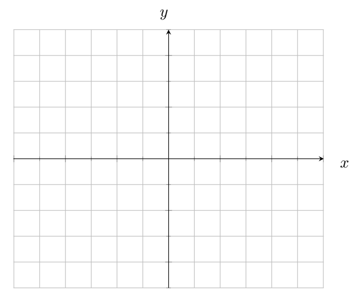
3. $f(x) = x^{\frac{3}{4}}$



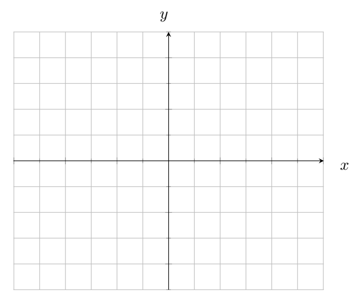
4. $f(x) = x^{\frac{5}{7}}$



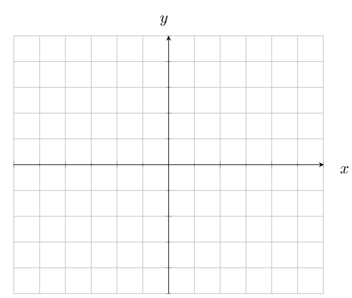
5. $f(x) = x^{\frac{6}{3}}$



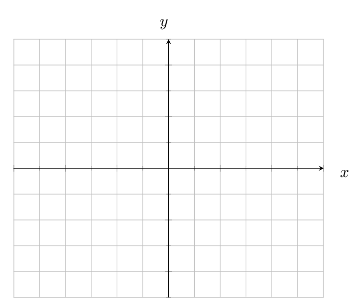
6. $f(x) = x^{\frac{4}{3}}$



7. $f(x) = x^{\frac{5}{4}}$



8. $f(x) = x^{\frac{7}{5}}$



Textbook Theorem 4.3. Let r and s be rational numbers. The following properties hold provided none of the computations results in division by 0 and either r and s have odd denominators or $x \geq 0$ and $y \geq 0$:

- **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$
- **Quotient Rules:** $\frac{x^r}{x^s} = x^{r-s}$ and $\left(\frac{x}{y}\right)^r = \frac{x^r}{x^r}$
- **Power Rule:** $(x^r)^s = x^{rs}$

Textbook Theorem 4.4. For real numbers a, b, h, and k and a rational number r with $a, b, r \neq 0$, the graph of $F(x) = a(bx-h)^r + k$ can be obtained from the graph of $f(x) = x^r$ by performing the following operations, in sequence:

1. add h to each of the x-coordinates of the points on the graph of f. This results in a horizontal shift to the right if $h > 0$ or left if $h < 0$.
   NOTE: This transforms the graph of $y = x^r$ to $y = (x-h)^r$.

2. divide the x-coordinates of the points on the graph obtained in Step 1 by b. This results in a horizontal scaling, but may also include a reflection about the y-axis if $b < 0$.
   NOTE: This transforms the graph of $y = (x-h)^r$ to $y = (bx-h)^r$.

3. multiply the y-coordinates of the points on the graph obtained in Step 2 by a. This results in a vertical scaling, but may also include a reflection about the x-axis if $a < 0$.
   NOTE: This transforms the graph of $y = (bx-h)^r$ to $y = a(bx-h)^r$.

4. add k to each of the y-coordinates of the points on the graph obtained in Step 3. This results in a vertical shift up if $k > 0$ or down if $k < 0$.
   NOTE: This transforms the graph of $y = a(bx-h)^r$ to $y = a(bx-h)^r + k$.

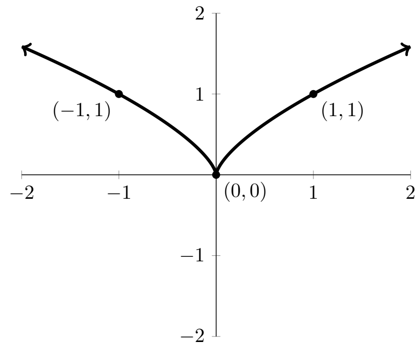
In English 4.1. Given a function of the form $F(x) = a(bx-h)^r + k$, write down the parent function $P(x) = x^r$. Identify *sample points* from the parent function, then do the following in order:

- Add h to every x-value.
- Divide all x-values by b.
- Multiply all y-values by a.
- Add k to every y-value.

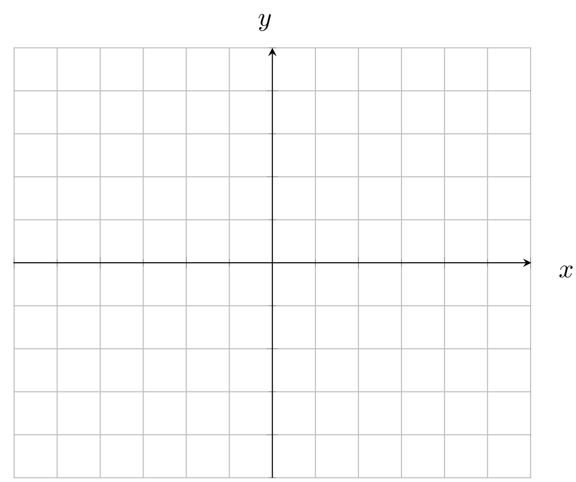
Graph these new points, then trace the parent function.

Picking Sample Points: Choosing sample points from a power function is more difficult than what we have seen before. In many problems, sample points will actually be provided to you directly. In the case that they are not, choose the simplest ones (like $(0,0)$ or some combination of $(\pm 1, \pm 1)$).

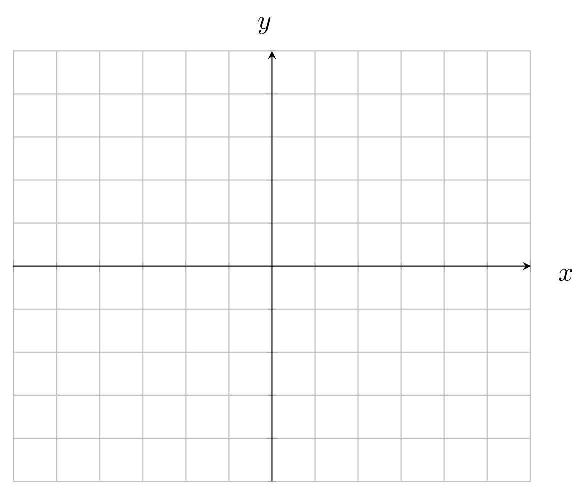
The graph of $f(x) = x^{\frac{2}{3}}$ looks as follows with the provided sample points:



1. Use $f(x)$ as provided above to graph $F(x) = (x-2)^{\frac{2}{3}} - 1$



2. Use $f(x)$ as provided above to graph $F(x) = 3 - x^{\frac{2}{3}}$



Irrational Powers: So far we have only discussed rational (fractional) values of p for a power function. However, what happens if we define a function like $f(x) = x^{\pi}$? Accurately evaluating functions like this is beyond the scope of this course, but we can generalize it using a few rules.

- If $0 < p < 1$ the function is concave down, if $p > 1$ the function is concave up
- If p is irrational (cannot be written as a fraction), the domain is $[0, \infty)$[2]

Irrational powers are less common in a course like this, so you don't need to worry too much about them.

Problem Solving Tip 1. To find the domain of a power function, convert it to a root function and apply the methods outlined in Section 4.1.

3. **Worked Example:** Find the domain: $f(x) = x^{\frac{2}{3}}(x-7)^{\frac{1}{3}}$

Scan the QR code for a video solution.

4. Find the domain: $g(t) = 2t(t+3)^{-\frac{1}{3}}$

[2]This is done for convenience, as things get complicated really quickly if we try to approximate an irrational value using fractions.

5. Find the domain: $f(x) = x^{\frac{3}{2}}(x-7)^{\frac{1}{3}}$

6. Find the domain: $g(t) = t^{\frac{3}{2}}(t-2)^{-\frac{1}{2}}$

7. Find the domain: $f(x) = x^{0.4}(3-x)^{0.6}$