4.3 | Equations and Inequalities involving Power Functions

Solving equations involving power functions is slightly more involved than what we have seen before. We will need to be a bit clever with some solutions, and we also need to be careful because certain power functions (even root functions) have restricted domains which can lead to extraneous solutions. First, let's simply cover the general method.

Solving Equations Involving Power Functions: The following steps outline how to solve an equation with a power function, but may not be a comprehensive list.

- **OPTIONAL**: Convert root functions to power functions, or vice versa to help you visualize the problem better.
- Where possible, merge similar powered terms together.
- Undo a power by raising the equation to an exponent that will remove one or more powers from the equation. Repeat this step until *all* powers are removed from the equation.
- Solve the equation algebraically.
- Check solutions (even root functions have a restricted domain).

Due to the slightly increased complexity of solving power functions in equations, let's expand on some of these steps to make things clearer.

Merging Terms Under a Single Power: After we isolate all terms of the equation to one side, it is a good idea to try and merge everything under a single root. While this isn't always possible, the following theorem from the previous section can be a great help.

> **Textbook Theorem 4.3.** Let r and s be rational numbers. The following properties hold provided none of the computations results in division by 0 and either r and s have odd denominators or $x \geq 0$ and $y \geq 0$:
>
> - **Product Rules:** $x^r x^s = x^{r+s}$ and $(xy)^r = x^r y^r$
> - **Quotient Rules:** $\dfrac{x^r}{x^s} = x^{r-s}$ and $\left(\dfrac{x}{y}\right)^r = \dfrac{x^r}{x^r}$
> - **Power Rule:** $(x^r)^s = x^{rs}$

Undoing Roots/Powers: The meaning of *undoing* a power is just finding a way to simplify the power (often to 1, or another "nice" number). For example, the equation $\sqrt{x+1} = 2$, while simple, can be made even easier to solve by *squaring both sides*. Upon doing this, see that $\left(\sqrt{x+1}\right)^2 = 2^2$ reduces to $x+1=4$ and we can easily see that $x = 3$. Of course, you might be able to guess that $x = 3$ just by looking at the original equation, however for increasingly difficult equations, this method becomes increasingly powerful.

Problem Solving Tip 1. If you want to undo an equation raised to a power of the form $\frac{m}{n}$, try raising the equation to the power $\frac{n}{m}$. This will cause the power to reduce to 1.

Another theorem that is good to be familiar with from previous sections, is the following:

Textbook Theorem 4.2. Some Useful Properties of Radicals: Suppose $\sqrt[n]{x}$, $\sqrt[n]{a}$ and $\sqrt[n]{b}$ are real numbers.
Simplifying nth powers and nth roots:

- $\left(\sqrt[n]{x}\right)^n = x$
- if n is odd, then $\sqrt[n]{x^n} = x$
- if n is even, then $\sqrt[n]{x^n} = |x|$

Root Functions Preserve Inequality: if $a \leq b$, then $\sqrt[n]{a} \leq \sqrt[n]{b}$.

Checking Solutions: When we go through the process of solving equations inside root functions, we sometimes simplify a root outside of a function, and then begin solving a slightly different equation than what we had before. In this process, we may come across solutions that don't work in the context of the original equation. These solutions are called *extraneous solutions*. Once you have solved the equation and have one or more solutions present, make sure you check each one and remove any solutions which lead to an impossible situations. Here are two of the most common situations that can lead to an impossibility.

- Even root functions cannot have a negative input. Example: The equation $\sqrt{-1} = x$ has no solution.
- Even root functions cannot have a negative output. Example: The equation $\sqrt{x} = -1$ has no solution.

1. **Worked Example:** Solve the equation: $x + 1 = (3x + 7)^{\frac{1}{2}}$

Scan the QR code for a video solution.

2. Solve the equation: $2x + 1 = (3 - 3x)^{\frac{1}{2}}$

3. Solve the equation: $3t + (6 - 9t)^{0.5} = 2$

4. Solve the equation: $x^{-1.5} = 8$

5. Solve the equation: $t^{\frac{2}{3}} = 4$

6. Solve the equation: $(t-2)^{\frac{1}{2}} + (t-5)^{\frac{1}{2}} = 3$

7. Solve the equation: $(2x+1)^{\frac{1}{2}} = 3 + (4-x)^{\frac{1}{2}}$

Problem Solving Tip 2. u substitution

This method simplifies equations by replacing a recurring term by a new variable (traditionally u). You can solve for this new variable, then substitute the replaced term back in and find solutions to the original equation.

For example: $(3x-2)^2 - 5(3x-2) + 6$ might look like it will contain a painstakingly long amount of time to expand out and solve, however we can save time by defining the term $u = 3x - 2$. Now we can rewrite the equation using this new variable: $u^2 - 5u + 6$. This is a simple quadratic, which we can factor and solve to find that $u = 2$ and $u = 3$. Now that we have isolated our new variable u, we can replace it with our original substitution ($u = 3x - 2$) and solve the two equations $3x - 2 = 2$ and $3x - 2 = 3$. Solving these equations gives the solutions $x = \frac{4}{3}$ and $x = \frac{5}{3}$ which are indeed the solutions to the original equation.

8. Solve the equation: $2t^{\frac{2}{3}} = 6 - t^{\frac{1}{3}}$

9. Solve the equation: $2t^{\frac{1}{3}} = 1 - 3t^{\frac{2}{3}}$

Solving an Inequality involving a Power Function: The following steps outline how to solve an inequality with a power function, but may not be a comprehensive list. Certain inequalities will require a more careful consideration of the function and will not fit directly into these steps.

- Reduce **odd** powers in the equation.
- Move all terms to one side of the inequality using only addition and subtraction.
- Factor where possible **OR** find common denominators and merge fractions.
- Construct a sign diagram (check for domain restriction) and interpret.

Reduce odd powers: If we raise the equation to any even power, the inequality relation might not be preserved. For example: it is true that $-2 < 1$, but if we raise both sides to a power of 2 we see that $(-2)^2 < (1)^2$ simplifies to $4 < 1$ which is an untrue statement. For this reason, we should avoid raising to an even power at all costs, as our answers may not end up being correct. We can however, raise to an odd power. This will preserve inequality and odd powers maintain the negative symbol in an operation. Use the reduction of odd powers to your advantage to simplify an inequality.

Move terms to one side of the equation: You might recall that multiplying or dividing by a negative value makes the inequality sign flip. It might be tempting in some problems to try and divide a term to move it to another side of the equation, however we should avoid doing this. Since we are dealing with variables, there are often instances where a term can be positive for some values, and negative for others. Therefore there is no clear way to flip the inequality sign. It is best to move terms to one side of the equation using only addition and subtraction so that the inequality relationship is preserved.

Factor or find common denominators: If you are able to reduce all the powers from a function, treat it like a polynomial, and try factoring. Then you can construct a sign diagram based on the zeros of the factors. In instances where you have a negative power, you will want to convert the expression into a fraction, and then find a common denominator with other terms and merge everything into a rational function. From this point, you can solve for zeros and asymptotes and make a sign diagram using these values.

Sign diagrams of power functions: Sign diagrams will work similarly to how they did in previous sections, we will simply need to adjust certain aspects of the method to accommodate for power functions.

- **Even root functions have a *restricted* domain.** If we want to make an even root function a part of a sign diagram, you should immediately analyze the domain of the function. The sign diagram can only exist where the domain of the function exists, so we can start/end our number line depending on the domain restriction.
- **Generalizing to a linear or quadratic.** When we make our sign diagram, we are only concerned with capturing the positive and negative intervals of a factor, and we do not need to do an in depth analysis of the shape (concavity) of a power function. Therefore, we can generalize to two main types of lines, linear and quadratic. Before you do this, check to make sure the rational powers of a function are in lowest terms, and that the domain is not restricted.
  - **Power Functions that are "Linear"**: If $p(x) = x^{\frac{m}{n}}$ with both n and m odd, the function will mimic a linear.
  - **Power Functions that are "quadratic"**: If $p(x) = x^{\frac{m}{n}}$ with n odd and m even, the function will mimic a quadratic.

10. **Worked Example:** Solve the inequality: $10 - \sqrt{t-2} \leq 11$

Scan the QR code for a video solution

11. **Worked Example:** Solve the inequality: $t^{\frac{2}{3}} \leq 4$

Scan the QR code for a video solution

12. Solve the inequality: $\sqrt[3]{x} \leq x$

13. Solve the inequality: $(2-3x)^{\frac{1}{3}} > 3x$ (*Hint: the discriminant might be useful when analyzing the quadratic.*)

14. Solve the inequality: $(t^2-1)^{-\frac{1}{2}} \geq 2$

15. Solve the inequality: $3(x-1)^{\frac{1}{3}} + x(x-1)^{-\frac{2}{3}} \geq 0$

16. Solve the inequality: $\sqrt[3]{x^3 + 3x^2 - 6x - 8} > x + 1$

17. Solve the inequality: $t^{\frac{2}{3}} < t^{\frac{4}{3}} - 6$