MATH1300
Selected Challenge Problems
Volume I
SOLUTIONS

Precalculus Peer Assisted Learning

February 3, 2025

Solution Preface:

I make no guarantee that these solutions are free of typos or errors. I have done my best to ensure they are all correct; however, you should always double-check my solutions with another resource if you are unsure whether they are accurate.

Function graphs in this packet are generated by a computer and may not look exactly like the ones you draw. Remember, you are not a computer. The most important skill to develop is your ability to analyze key information about the graph and sketch a relatively accurate picture.

Roman

1.1

A Observe the following equation: $2xy = 4$.

i. Does this equation represent y as a function of x?
Yes

ii. If so, write the domain of the equation as set, if not, provide an example where it fails as a function.
$\{x \mid x \in \mathbb{R} \text{ and } x \neq 0\}$

B Observe the set of ordered pairs

$$\{(-3, 9), (1, 1), (3, 1), (0, 0), (-2, 4), (-3, 7), (4, 0)\}$$

i. Does the set of ordered pairs represent a function?
No

ii. If so, write the domain as a set, if not, provide an example where it fails as a function.
$f(-3) = 9 = 7$

C Observe the following data table.

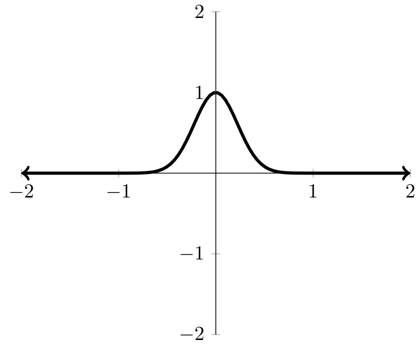
x	y
-3	3
-2	2
-1	1
0	0
1	1
2	2
3	3

i. Does the given table represent y as a function of x? Explain.
Yes

ii. Write the domain of the table as a set.
$\{-3, -2, -1, 0, 1, 2, 3\}$

iii. Write the range of the table as a set.
$\{0, 1, 2, 3\}$

D Observe the graph

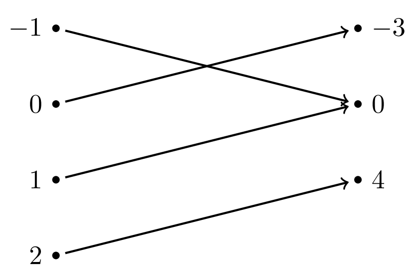


i. Does the graph represent a function? Explain.
   Yes, passes vertical line test.

ii. Write the domain of the graph using interval notation.
   $(-\infty, \infty)$

iii. Write the range of the graph using interval notation.
   $(0, 1]$

E Consider the function f as a mapping diagram shown:



i. Write the domain of f as a set.
   $\{-1, 0, 1, 2\}$

ii. Write the range of f as a set.
   $\{-3, 0, 4\}$

iii. Find $f(0)$ and solve $f(x) = 0$.
   $f(0) = -3$ and $f(x) = 0$ implies $x = -1$ or $x = 1$.

iv. Write f as a set of ordered pairs.
   $\{(-1, 0), (0, -3), (1, 0), (2, 4)\}$

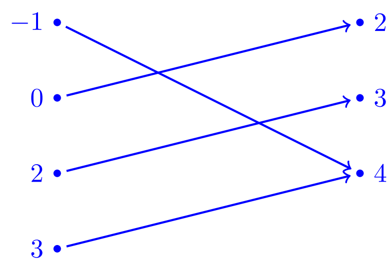
F Let $g = \{(-1,4),(0,2),(2,3),(3,4)\}$

i. Write the domain of g as a set.
$\{-1, 0, 2, 3\}$

ii. Write the range of g as a set.
$\{2, 3, 4\}$

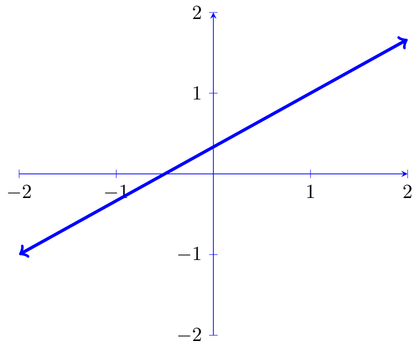
iii. Find $g(0)$ and solve $g(x) = 0$.
$g(0) = 2$ and $g(x) = 0$ has no solution.

iv. Create a mapping diagram for g.



1.2

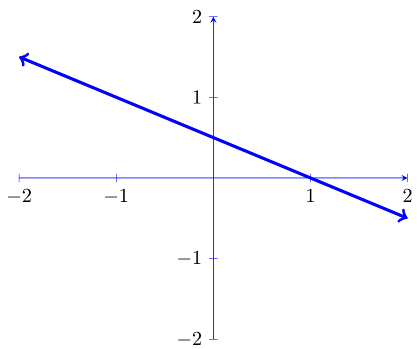
A Graph the function $h(t) = \frac{2}{3}t + \frac{1}{3}$.



i. What is the slope?
$\frac{2}{3}$

ii. State the axis intercepts, if they exist.
$\left(-\frac{1}{2}, 0\right), \left(0, \frac{1}{3}\right)$

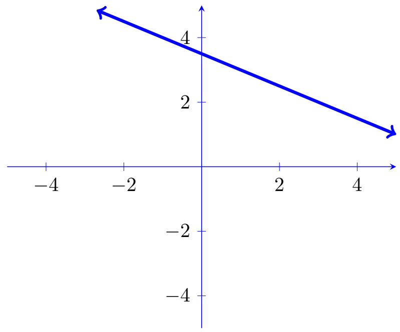
B Graph the function $j(w) = \dfrac{1-w}{2}$



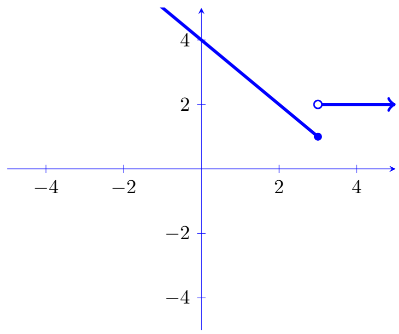
i. What is the slope?
$-\frac{1}{2}$

ii. State the axis intercepts, if they exist.
$\left(0, \frac{1}{2}\right), (1, 0)$

C Find the equation of the function that contains the points $(1, 3)$ and $(3, 2)$.



D Graph the piecewise function $f(x) = \begin{cases} 4 - x & \text{if } x \leq 3 \\ 2 & \text{if } x > 3 \end{cases}$

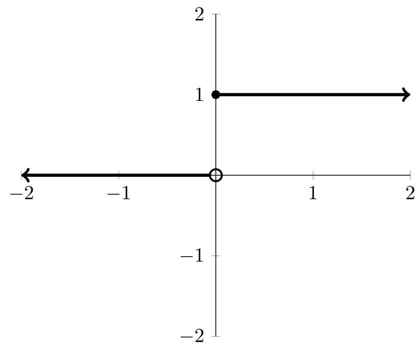


i. Write the domain in interval notation.
   $(-\infty, \infty)$

ii. Write the range in interval notation.
   $[1, \infty)$

iii. State the axis intercepts, if they exist.
   $(0, 4)$

E The <u>unit step function</u> is graphed below:



i. Write the equation $U(t)$ of the unit step function.

$$U(t) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 1 \end{cases}$$

ii. Write the domain of $U(t)$

$(-\infty, \infty)$

iii. Write the range of $U(t)$

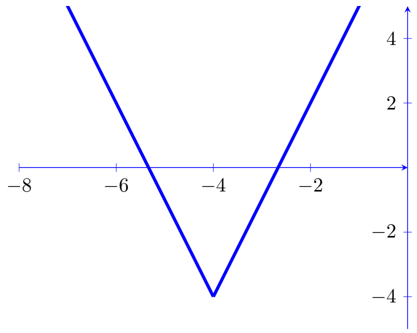
$\{0, 1\}$

F* Explain why the graph of a function $f(x)$ must have at most one y-intercept.

Assume $f(x)$ has more than one y-intercept. Draw a horizontal line on the y-axis, this line intersects the graph more than once, and thus it fails the vertical line test and is not a function.

1.3

A Graph the function $g(t) = 3|t + 4| - 4$

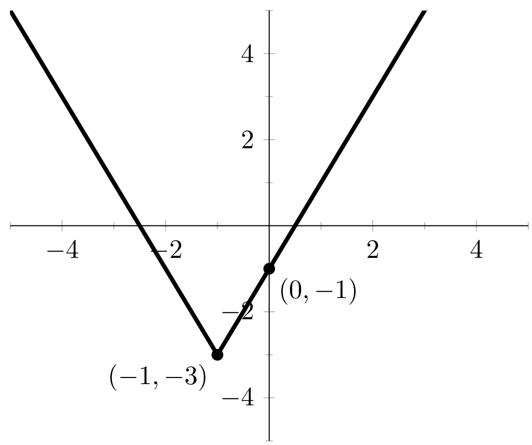


i. Write the domain of $g(t)$ in interval notation.
$(-\infty, \infty)$

ii. Write the range of $g(t)$ in interval notation.
$[-4, \infty)$

iii. State the axis intercepts, if they exist.
$(0, 8)$

B The graph of $F(x)$ is shown below:

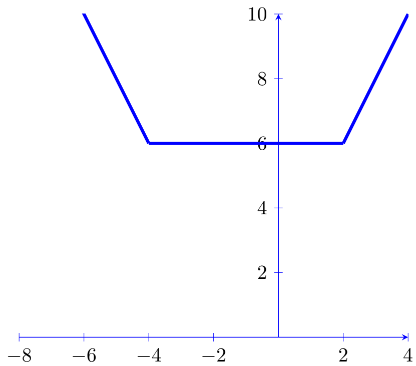


i. Write piecewise function definition of $F(x)$.
$F(x) = \begin{cases} -2x - 5 & \text{if } x < -1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$

ii. State the domain of $F(x)$.
$(-\infty, \infty)$

iii. State the range of $F(x)$.
$[-3, \infty)$

C Graph the function $g(x) = |t+4| + |t-2|$.



i. Write the domain of $g(x)$ using interval notation.
$(-\infty, \infty)$

ii. Write the range of $g(x)$ using interval notation.
$[6, \infty)$

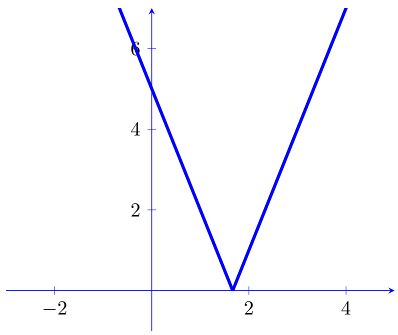
iii. State axis intercepts, if they exist.
$(0, 6)$

D Solve the equation $|3x-2| = |2x+7|$.

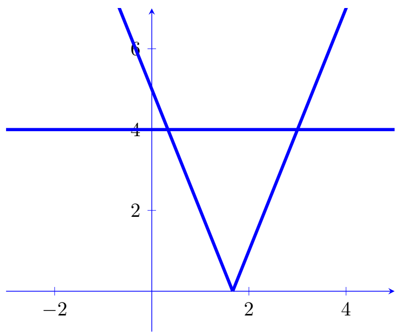
i. Write the solutions as a set.
$\{-1, 9\}$

E Given $f(x) = |3x - 5|$ and $g(x) = 4$

i. Graph $f(x)$.



ii. Graph $g(x)$ (on the same plot).



iii. Solve $f(x) \leq g(x)$. Write your answer in interval notation.
$\left[\frac{1}{3}, 3\right]$

F* Show that if d is a real number with $d > 0$, the solution to $|x - a| < d$ is the interval: $(a - d, a + d)$. That is, an interval centered at a with 'radius' d.

Proof. From the definition of absolute value we know that the distance between x and a must be less than d, we can rephrase this with the relationship $-d < x - a < d$. Adding a to both sides we obtain $-d + a < x < d + a$. With some rearranging we obtain $a - d < x < a + d$ which provides the solution interval $(a - d, a + d)$ for x. □

1.4

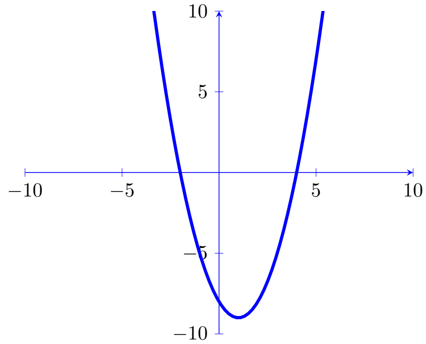
A Let $f(x) = x^2 - 2x - 8$

i. Complete the square on $f(x)$.
$f(x) = (x-1)^2 - 9$

ii. Write the vertex.
$(1, -9)$

iii. Find the axis intercepts.
$(-2, 0), (4, 0)$

iv. Graph $f(x)$.



B Let $h(t) = -3t^2 + 5t + 4$

i. Compute the discriminant of $h(t)$. How many real zeros does $h(t)$ have?
72, this means the function has two positive real roots.

ii. Find the zero(s) of $h(t)$ if they exist, write your solutions as a set.
$\left\{ \frac{5-\sqrt{73}}{6}, \frac{5+\sqrt{73}}{6} \right\}$

C Let $g(x) = x^2 - 3x + 9$

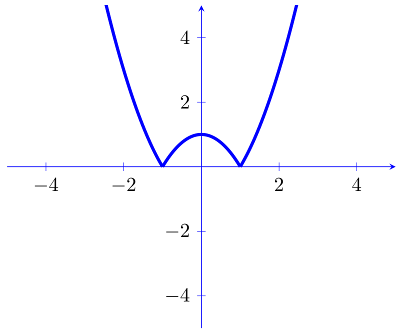
i. Is $g(x)$ factorable?
No

ii. If yes, write $g(x)$ in factored form. If not, explain why.
The discriminant of $g(x)$ is -27, which implies that the function has no real zeros. Therefore it is not factorable.

D Solve the inequality $3x^2 \leq 11x + 4$, write your answer in interval notation.
$\left[-\frac{1}{3}, 4\right]$

E Solve the inequality $5t + 4 \leq 3t^3$, write your answer in interval notation.
$\left(-\infty, \frac{5-\sqrt{73}}{6}\right] \cup \left[\frac{5+\sqrt{73}}{6}, \infty\right)$

F* Graph $f(x) = |1 - x^2|$.



2.1

A Let $g(x) = 3x^5 - 2x^2 + x + 1$

i. Identify the degree of $g(x)$.
   5

ii. Identify the leading coefficient of $g(x)$.
   3

iii. Identify the leading term of $g(x)$.
   $3x^5$

iv. Identify the constant term of $g(x)$.
   1

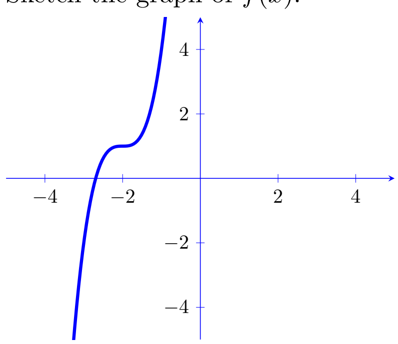
v. Write the end behavior of $g(x)$.
   as $x \to \infty, f(x) \to \infty$, as $x \to -\infty, f(x) \to -\infty$

B Let $f(x) = 3(x+2)^3 + 1$

i. Write the parent function $P(x)$ for $f(x)$.
   $P(x) = x^3$

ii. Pick three points from the parent function $P(x)$ and apply the transformations of $f(x)$ to write three points on the graph of $f(x)$.

iii. Sketch the graph of $f(x)$.



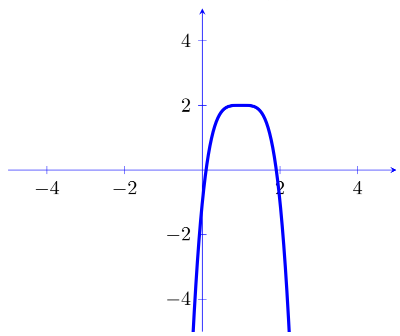
iv. State the domain and range of $f(x)$ using interval notation.
   Domain and Range both $(-\infty, \infty)$

C Let $f(x) = 2 - 3(x-1)^4$

i. Write the parent function $P(x)$ for $f(x)$.
$P(x) = x^4$

ii. Pick three points from the parent function $P(x)$ and apply the transformations of $f(x)$ to write three points on the graph of $f(x)$.

iii. Sketch the graph of $f(x)$.



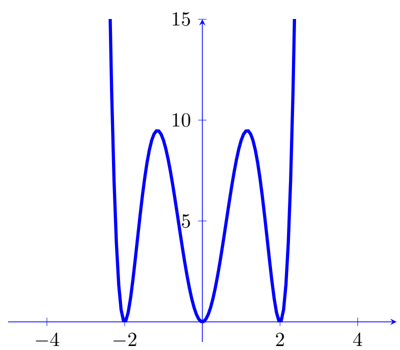
iv. State the domain and range of $f(x)$ using interval notation.
Domain: $(-\infty, \infty)$, Range: $(-\infty, 2]$

D Let $h(t) = t^2(t-2)^2(t+2)^2$

i. List all zeros of $h(t)$ and their corresponding multiplicities.
$t = -2_{m=2}, t = 0_{m=2}, t = 2_{m=2}$

ii. Write the end behavior of $h(t)$.
as $x \to \infty, f(x) \to \infty$, as $x \to -\infty, f(x) \to \infty$.

iii. Sketch a graph of the function $h(t)$.

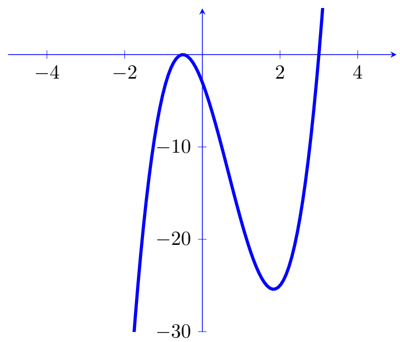


E Let $g(x) = (2x+1)^2(x-3)$

i. List all zeros of $g(x)$ and their corresponding multiplicities.
$x = -\frac{1}{2}_{m=2}, t = 3_{m=1}$

ii. Write the end behavior of $g(x)$.
as $x \to \infty, f(x) \to \infty$, as $x \to -\infty, f(x) \to -\infty$.

iii. Sketch a graph of the function $g(x)$.



F Let $f(x) = (x^2+1)(x-1)$

i. Determine analytically if $f(x)$ is even, odd, or neither.
neither

2.2

A Let $f(z) = 4z^3 + 2z - 3$ and $g(z) = z - 3$

i. Compute $f(z)/g(z)$.
$(4z^3 + 2z - 3) \div (z - 3) = (4x^2 + 12z + 38)\ R11$

ii. Write $f(z)$ as an expression involving $g(z)$, a quotient, and remainder (if it exists).
$(4z^3 + 2z - 3) = (z - 3)(4z^2 + 12z + 38) + 11$

B Let $f(x) = 2x^3 - x + 1$ and $g(x) = x^2 + x + 1$

i. Compute $f(x)/g(x)$.
$(2x^3 - x + 1) \div (x^2 + x + 1) = (2x - 2)\ R(3 - x)$

ii. Write $f(x)$ as an expression involving $g(x)$, a quotient, and remainder (if it exists).
$(2x^3 - x + 1) = (2x - 2)(x^2 + x + 1) + (3 - x)$

C Let $a(x) = x^4 - 6x^2 + 9$ and $b(x) = (x - \sqrt{3})$

i. Compute $a(x)/b(x)$.
$(x^4 - 6x^2 + 9) \div (x - \sqrt{3}) = (x^3 + \sqrt{3}x^2 - 3x - 3\sqrt{3})\ R0$

ii. Write $a(x)$ as an expression involving $b(x)$, a quotient, and remainder (if it exists).
$x^4 - 6x^2 + 9 = (x^3 + \sqrt{3}x^2 - 3x - 3\sqrt{3})(x - \sqrt{3})$

D Let $g(z) = z^3 + 2z^2 - 3z - 6$ be a polynomial function with a known real zero of $c = -2$

i. Find the remaining real zeros of $g(z)$
$z = -2, \sqrt{3}, -\sqrt{3}$

E Let $x^3 - 6x^2 + 11x - 6$ be a polynomial function with a known real zero of $c = 1$

i. Find the remaining real zeros of $g(z)$
$x = 1, 2, 3$

F* Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$ be a polynomial function with the property that $a_n + a_{n-1} + \cdots + a_1 + a_0 = 0$. (That is, the sum of the coefficients and the constant term is 0.)

i. Show that $(x - 1)$ is a factor of $f(x)$.

Proof.

If $(x - 1)$ is a factor then $f(1) = 0$. Plug in $x = 1$ to $f(x)$ to obtain $f(1) = (a_n)1 + (a_{n-1})1 + \cdots + (a_1)1 + (a_0)1$ which simplifies to $a_n + a_{n-1} + \cdots + a_1 + a_0$. By our assumption, $a_n + a_{n-1} + \cdots + a_1 + a_0 = 0$. So $f(1) = 0$ and thus $(x - 1)$ is a factor of f. □