5.2 | Function Arithmetic

In this section we introduce definitions and theorems that generalize the way in which we can treat functions as their own objects and do arithmetic with them. Many of these definitions/theorems turn out exactly as we would expect, but it is still useful to write them down as a reference for what we are allowed to do with functions.

Definitions of Arithmetic: Suppose f and g are functions and x is in both the domain of f and the domain of g.

- **Sum**: $f + g$ is defined by the formula:

$$(f + g)(x) = f(x) + g(x)$$

- **Difference**: $f - g$ is defined by the formula:

$$(f - g)(x) = f(x) - g(x)$$

- **Product**: fg is defined by the formula:

$$(fg)(x) = f(x)g(x)$$

- **Quotient**: $\frac{f}{g}$ is defined by the formula:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

  provided that $g(x) \neq 0$.

In addition to the basic arithmetic operations, the arithmetic rules follow as outlined in the following theorem.

Textbook Theorem 5.1. Suppose f, g, and h are functions.

- **Commutative Law of Addition**: $f + g = g + f$
- **Associative Law of Addition**: $(f + g) + h = f + (g + h)$
- **Additive Identity**: The function $Z(x) = 0$ satisfies: $f + Z = Z + f = f$ for all functions f.
- **Additive Inverse**: The function $F(x) = -f(x)$ for all x in the domain of f satisfies:

$$f + F = F + f = Z$$

- **Commutative Law of Multiplication**: $fg = gf$
- **Associative Law of Multiplication**: $(fg)h = f(gh)$
- **Multiplicative Identity**: The function $I(x) = 1$ satisfies $fI = If = f$ for all functions f.
- **Multiplicative Inverse**: If $f(x) \neq 0$ for all x in the domain of f, then $F(x) = \dfrac{1}{f(x)}$ satisfies:

$$fF = Ff = I$$

- **Distributive Law of Multiplication over Addition**: $f(g + h) = fg + fh$

1. Given $f(x) = 3x + 1$ and $g(t) = 4 - t$ compute the following:

   (a) $(f + g)(2)$

   (b) $(f - g)(-1)$

2. Given $f(x) = x^2$ and $g(t) = -2t + 1$ compute the following:

   (a) $(g - f)(1)$

   (b) $(fg)\left(\frac{1}{2}\right)$

3. Given $f(x) = x^2 - x$ and $g(t) = 12 - t^2$ compute the following:

   (a) $\left(\frac{f}{g}\right)(0)$

   (b) $\left(\frac{g}{f}\right)(-2)$

4. Let f be the function defined by

$$f = \{(-3,4),(-2,2),(-1,0),(0,1),(1,3),(2,4),(3,-1)\}$$

and let g be the function defined by

$$g = \{(-3,-2),(-2,0),(-1,-4),(0,0),(1,-3),(2,1),(3,2)\}$$

compute the indicated value if it exists:

(a) $(f-g)(2)$

(b) $(gf)(-3)$

(c) $(g+f)(1)$

(d) $\left(\frac{f}{g}\right)(-1)$

Difference Quotient: A difference quotient[1] is a formula that helps us measure the amount of change in a function at a given point. A common formula for difference quotients is:

$$\frac{\Delta[f(x)]}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$

with $h \neq 0$. Here, we read the Greek character Δ as "change." So $\Delta[f(x)]$ can be interpreted as "change in $f(x)$." We measure this change by adding a value to x often called h. So $x + h$ is a new value which differs from x by h amount. Then we compute the *function value* at this point $f(x + h)$, and then we compute the function value at the original point (just x) $f(x)$. We can then measure the average amount of change between these points by subtracting $f(x)$ from $f(x+h)$ and then averaging over h. Putting this altogether yields the formula stated above.[2]

There are variations on the difference quotient formula. For one, sometimes we will be provided the value of x in advance. For example if we want to compute when $x = 2$ our formula would look like:

$$\frac{f(2+h) - f(2)}{h}$$

Alternatively, some formulas use Δx instead of h to emphasize that h is meant to represent an amount of change in x. You can treat "Δx" as a separate variable entirely from x (as opposed to reading it as Δ *times* x). This formula would look like:

$$\frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Solving Difference Quotients: To solve a difference quotient, we will simply follow the formula provided. If you are given a function f and you are trying to compute $f(x + h)$, replace every instance of x (or another variable used) in the function f, with $(x + h)$. Then simplify the new function. For example: if $f(x) = x + 3$, then $f(x + h) = (x + h) + 3 = x + h + 3$.

> **Problem Solving Tip 1.** When solving a difference quotient, the algebra can get very lengthy, and keeping everything inside a fraction can be difficult. It may be helpful to *separately* write out the definition of $f(x+h)$ and *fully simplify it* before writing it in the formula.

> **Problem Solving Tip 2.** In the formula for difference quotients, be very careful with your minus sign in between $f(x+h)$ and $f(x)$. It can help to write the expanded form of $f(x)$ in large brackets to remind yourself that the minus sign needs to be fully distributed throughout the entire function $f(x)$.

[1] Sometimes called a Newton Quotient.

[2] Difference quotients as a formula might not make a lot of sense in the context of this course. The skill of finding difference quotients is one that will make a return during the study of calculus. A difference quotient can be used to measure the *instantaneous rate of change* by analyzing what happens as the value of h grows smaller and smaller, getting as close to 0 as possible. This can be visualized as the *slope of a curve at a point*. For those who are more interested in this phenomenon, please see the following YouTube video `The paradox of the derivative | Chapter 2, Essence of calculus`.

For the following practice problems, use the difference quotient formula: $\dfrac{f(x+h)-f(x)}{h}$.

5. **Worked Example:** Compute the difference quotient: $f(x) = 2x - 5$

Scan the QR code for a video solution.

6. Compute the difference quotient: $f(x) = -3x + 5$

7. Compute the difference quotient: $f(x) = 6$

8. Compute the difference quotient: $f(x) = 3x^2 - x$

9. Compute the difference quotient: $f(x) = x - x^2$

10. Compute the difference quotient: $f(x) = ax^2 + bx + c$ where $a \neq 0$.