5.3 | Function Composition

When we worked with difference quotients, we saw how we can change the input values of a function by modifying (x) to be $(x+h)$. This changed every place that x showed up in our function definition to be replaced by $x+h$. Now, we take this idea even further. What do we do if we want to define the input of a function depending on the output of another function? This leads to an idea called function composition.

Function Composition: When we want to evaluate the input value x of $f(x)$ at the output value of $g(x)$, we can write $f(g(x))$. This is called the composition of f and g. The shorthand notation for this is to use a "$\circ$" to indicate when a function is composed with another. This notation looks like:

$$(f \circ g)(x) = f(g(x))$$

Computing Composition at a point: If we want to compute $(f \circ g)(c)$ where c is a defined value, we can do the following:

- **(Optional)** Rewrite $(f \circ g)(x)$ as $f(g(x))$. This can often help to better visualize what is being asked.
- Compute $g(c)$. Record the output.
- Plug the output of $g(c)$ into $f(x)$. Record the output.

1. Let $f(x) = x^2$ and $g(t) = 2t + 1$. Evaluate $(g \circ f)(0)$.

2. Let $f(x) = 4 - x$ and $g(t) = 1 - t^2$. Evaluate $(f \circ g)(-1)$.

Simplifying Composition: If we want to simplify $(f \circ g)(x)$, we can do the following:

- **(Optional)** Rewrite $(f \circ g)(x)$ as $f(g(x))$. This can often help to better visualize what is being asked.
- Identify the variable used in $f(x)$.
- Replace every instance of the variable x occurring in $f(x)$ with the entire function $g(x)$.
- Simplify the result.

Composition has some properties similar to other operations that we see in mathematics. The following theorem highlights the associative and identity properties of composition.

Textbook Theorem 5.2. Properties of Function Composition: Suppose f, g, and h are functions.

- **Associative Law of Composition:** $h \circ (g \circ f) = (h \circ g) \circ f$, provided the composite functions are defined.
- **Composition Identity:** The function $I(x) = x$ satisfies $I \circ f = f \circ I = f$ for all functions, f.

3. **Worked Example:** Let $f(x) = 2x + 3$ and $g(t) = t^2 - 9$. Simplify $(g \circ f)(x)$.

Scan the QR code for a video solution.

4. Let $f(x) = x^2 - x + 1$ and $g(t) = 3t - 5$. Simplify $(f \circ g)(t)$.

5. Let $f(x) = x^2 - 4$. Compute $(f \circ f)(x)$.

6. Let $f(x) = 3x - 5$ and $g(t) = \sqrt{t}$. Simplify both $(g \circ f)(x)$ and $(f \circ g)(t)$.

7. Let $f(x) = x^2 + 4x + 1$ and $g(t) = \sqrt{t + 3}$. Simplify $(g \circ f)(x)$.