5.6 | Inverse Functions

Inverse Functions: Two functions f and g are said to be inverses and considered invertible if the following relationship holds:

$$(g \circ f)(x) = x = (f \circ g)(x)$$

Textbook Theorem 5.10. Properties of Inverse Functions: Suppose f is an invertible function.

- There is exactly one inverse function for f, denoted f^{-1} (read 'f-inverse')
- The range of f is the domain of f^{-1} and the domain of f is the range of f^{-1}
- $f(a) = c$ if and only if $a = f^{-1}(c)$
  NOTE: In particular, for all y in the range of f, the solution to $f(x) = y$ is $x = f^{-1}(y)$.
- (a, c) is on the graph of f if and only if (c, a) is on the graph of f^{-1}
  NOTE: This means the graph of $y = f^{-1}(x)$ is the reflection of the graph $y = f(x)$ across $y = x$.
- f^{-1} is an invertible function and $(f^{-1})^{-1} = f$.

1. Verify that the following functions are inverses: $f(x) = 2x + 7$ and $g(x) = \dfrac{x - 7}{2}$

2. Verify that the following functions are inverses: $f(x) = \dfrac{5 - 3x}{4}$ and $g(x) = -\dfrac{4}{3}x + \dfrac{5}{3}$

3. Verify that the following functions are inverses: $f(x) = \dfrac{5}{t-1}$ and $g(x) = \dfrac{t+5}{t}$

4. Verify that the following function is its own inverse: $f(x) = \dfrac{t}{t-1}$

One-to-one: A function f is said to be one-to-one if whenever $f(a) = f(b)$, then $a = b$.[1] This definition is very useful as it ties together well with the following theorem.

Textbook Theorem 5.11. Equivalent Conditions for Invertibility:
For a function f, either all of the following statements are true or none of them are:

- f is invertible.
- f is one-to-one.
- The graph of f passes the Horizontal Line Test.[a]

[a] i.e., no horizontal line intersects the graph more than once.

How to find an inverse: If we want to find the inverse of a function directly, we can use the following steps.

- Write $y = f(x)$.
- Switch all instances of y with x and all instances of x with y.
- Solve for $x = f(y)$ for y. This new equation is $y = f^{-1}(x)$.

[1] The fancy math term for this is that f is *injective*.

5. **Worked Example:** Show that the following function is one-to-one and find its inverse: $f(x) = 6x - 2$

Scan the QR code for a video solution.

6. Show that the following function is one-to-one and find its inverse: $g(t) = \dfrac{t-2}{3} + 4$

7. Show that the following function is one-to-one and find its inverse: $f(x) = \sqrt{3x-1} + 5$

8. Show that the following function is one-to-one and find its inverse: $f(x) = 2 - \sqrt{x-5}$

9. Show that the following function is one-to-one and find its inverse: $g(t) = \dfrac{3}{4-t}$

10. Show that the following function is one-to-one and find its inverse: $f(x) = \dfrac{2x-1}{3x+4}$