6.1 | Exponential Functions

Exponential Functions: An exponential function is of the form

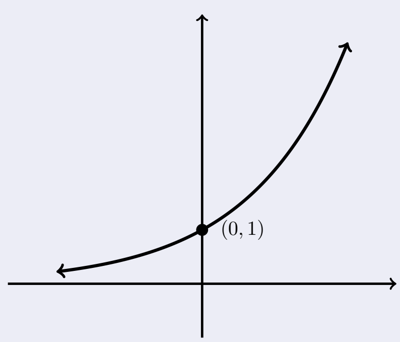
$$f(x) = b^x$$

where b is a real number, $b > 0$ and $b \neq 1$. We refer to the number b as the *base*.

Textbook Theorem 6.1. Properties of Exponential Functions: Suppose $f(x) = b^x$.

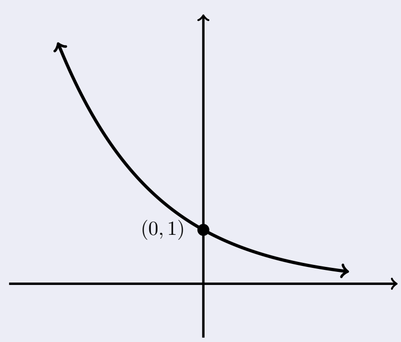
- The domain of f is $(-\infty, \infty)$ and the range of f is $(0, \infty)$.
- $(0, 1)$ is on the graph of f and $y = 0$ is a horizontal asymptote to the graph of f.
- f is one-to-one, continuous, and smooth[a].

- If $b > 1$:
  - f is always increasing
  - As $x \to -\infty$, $f(x) \to 0^+$
  - As $x \to \infty$, $f(x) \to \infty$
  - The graph of f resembles:



$y = b^x, b > 1$

- If $0 < b < 1$:
  - f is always decreasing
  - As $x \to -\infty$, $f(x) \to \infty$
  - As $x \to \infty$, $f(x) \to 0^+$
  - The graph of f resembles:



$y = b^x, 0 < b < 1$

[a] Recall that this means the graph of f has no sharp turns or corners.

Textbook Theorem 6.2. (Algebraic Properties of Exponential Functions) Let $f(x) = b^x$ be an exponential function ($b > 0, b \neq 1$) and let u and w be real numbers.

- **Product Rule:** $f(u+w) = f(u)f(w)$. In other words, $b^{u+w} = b^u b^w$
- **Quotient Rule:** $f(u-w) = \dfrac{f(u)}{f(w)}$. In other words, $b^{u-w} = \dfrac{b^u}{b^w}$
- **Power Rule:** $(f(u))^w = f(uw)$. In other words, $(b^u)^w = b^{uw}$

In English 6.2. (Algebraic Properties of Exponents) For any base b and real numbers x and y:

- $b^x b^y = b^{x+y}$
- $\dfrac{b^x}{b^y} = b^{x-y}$
- $(b^x)^y = b^{xy}$

Problem Solving Tip 1. Memorize the rules from Theorem 6.2.

***e*:** In this section we introduce a new mathematical constant. This is the number we refer to as e. There are multiple names[1] given to e, though it is important enough that it almost always is understood to denote the following value:

$$e \approx 2.71828182845904523536028747135 2\ldots$$

e is transcendental[2], its numbers go on forever with no determinable pattern. Another famous transcendental number you may be familiar with is π. As a constant, e is important to our discussion of logarithms as e represents a natural rate of growth[3], and shows up in many instances when we want to model things like compound interest, population changes, and even probability.

Fun Facts About *e*:

- e has digits repeating forever, but modern computers make it feasible to find lots of these digits. In 2020 the record was set for finding 31 415 926 535 897 digits of e!
- The function $f(x) = e^x$ has a special property that at any given point on the curve, the slope of the function at that point (how much the function is growing at the point) is equal to the value of e^x itself! $f(x) = e^x$ is the only function that has this property.
- e, along with the constants π and i (where $i = \sqrt{-1}$) come together to form what is often called the most beautiful mathematical equation: $e^{\pi i} + 1 = 0$.

[1] Some popular ones are: Euler's Number and Napier's Constant, though the number was discovered by neither of these mathematicians, but instead by Jacob Bernoulli.

[2] The specific mathematical definition of transcendental is that e cannot be written as the solution to a polynomial. For example, some numbers are irrational, like $\sqrt{2} \approx 1.414$. This number has digits that repeat forever and cannot be written as a fraction, however the polynomial $r(x) = x^2 - 2$ provides $\sqrt{2}$ as a solution. There is no polynomial we can write down that will yield e as its solution.

[3] The context in which e was discovered arises from a problem about compound interest. We might start by asking: how much money do we make with a \$1 investment with 100% interest over 1 year? We would make \$1 and have a total of \$2. Then we ask, do we make more money if we invest \$1 over two 6-month intervals, with each interval giving a 50% return? We can calculate this and find we end up with \$2.25. If we take this to the extreme, both increasing the number of intervals throughout the year, and decreasing the percentage of return. How much money might we make with an infinite number of intervals throughout the year, but with infinitely small return on each interval? You would end up with \$$e$.

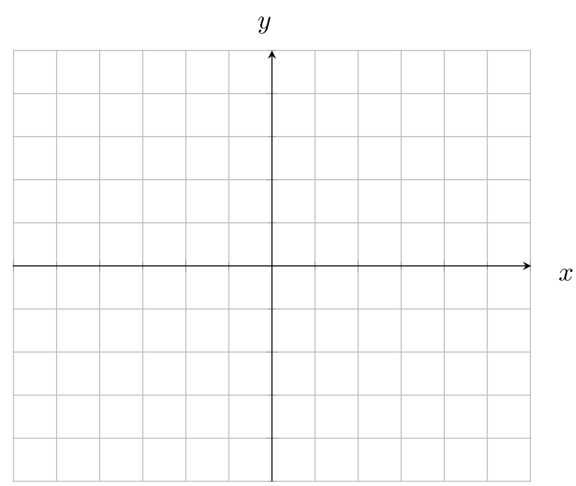
Function Transformations: To graph exponential functions, we can apply theorem 5.8 from section 5.4. Recall that this theorem puts an arbitrary function $f(x)$ into the form $af(bx-h)+k$. Our exponential functions look like $f(x)=b^x$, so we can apply the theorem by looking for values in the following context (we will bold the letters from theorem 5.8 to help distinguish the two b's present.):

$$F(x)=\mathbf{a}b^{\mathbf{b}x-\mathbf{h}}+\mathbf{k}$$

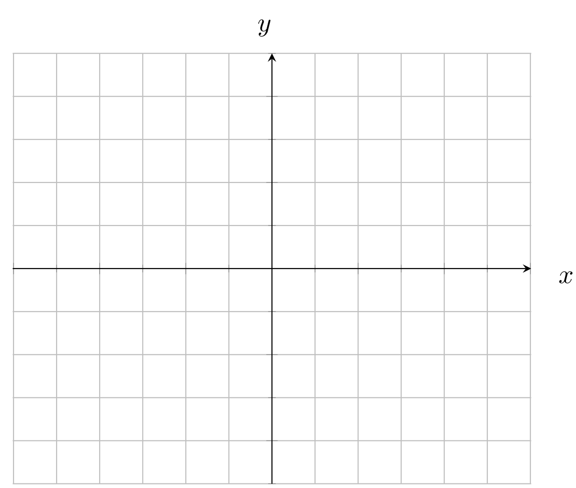
Choosing Sample Points: When choosing sample points for an exponential function, I recommend the following three where b is the value of the base present in the function $f(x)=b^x$.

$$\left\{\left(-1,\frac{1}{b}\right),(0,1),(1,b)\right\}$$

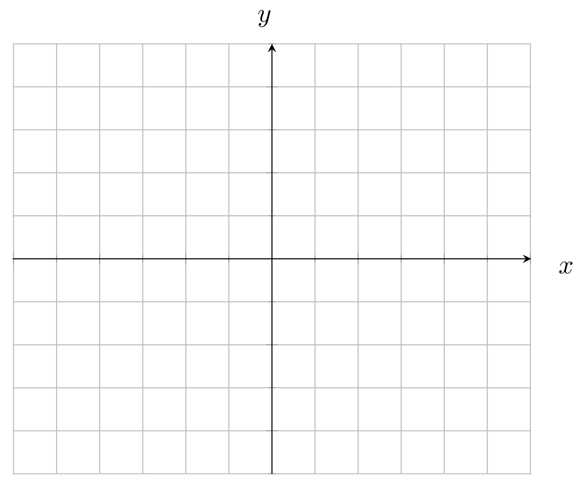
1. Sketch the graph of the function using transformations: $f(x)=2^x-1$



2. Sketch the graph of the function using transformations: $f(x)=3^{-x}+2$



3. Sketch the graph of the function using transformations: $f(x) = 10^{\frac{x+1}{2}} - 20$



4. Sketch the graph of the function using transformations: $f(x) = 8 - e^{-x}$