7.1 | The Radian Measures of Angles

If you have ever wondered, "why are there 360° in a circle," you are asking a very good question. The truth is that the number 360 is very arbitrary, and not mathematically justified.[1] Mathematicians recognized this arbitrary value, and have come up with an alternative system for measuring angles. Before we introduce this new system, allow us to review some key concepts.

Vocabulary of Circles: Circles have a few key parts that we need to be familiar with.

- **Circumference:** The circumference of a circle is the distance around the edge of the circle. It can be thought of as the perimeter.
- **Diameter:** The diameter is the distance across the circle that passes through the center.
- **Radius:** The radius of a circle is the distance from the center to the edge. The radius is exactly one half of the distance of the diameter.

The Relationship Between Circumference and Diameter: You may be familiar with the following formula. If we are given a circle and let C be the circumference and d be the diameter, the following relationship holds:

$$\pi = \frac{C}{d}$$

π is a real number that we assign to the value of $\frac{C}{d}$. We use the Greek letter π to represent this value as the number itself is transcendental[2], the digits go on forever, with no pattern. More importantly, this relationship holds for *any* circle whatsoever. We could divide the circumference and diameter of the Earth, or divide the circumference and diameter of the pupil in your eye, both instances will give us π.[3] We will use this important property of circles to define a new system to measure angles.

Redefining the Formula: Let the radius of a circle be represented by r. For any circle, the diameter is twice the radius, this leads to the first fact: $d = 2r$. We can substitute this value into our formula for π to obtain: $\pi = \frac{C}{2r}$. Finally, we can multiply over the 2 to obtain another very important relationship:

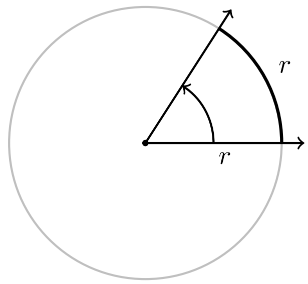
$$2\pi = \frac{C}{r}$$

[1] The historical reason, is that the number 360 comes from ancient Babylonian astronomers. The Babylonians recognized that 360 has many divisors, which makes it useful for calculations.

[2] The specific mathematical definition of transcendental is that π cannot be written as the solution to a polynomial. For example, some numbers are irrational, like $\sqrt{2} \approx 1.414$. This number has digits that repeat forever and cannot be written as a fraction, however the polynomial $r(x) = x^2 - 2$ provides $\sqrt{2}$ as a solution. There is no polynomial we can write down that will yield π as its solution.

[3] The Earth is not a perfect sphere/circle, your eye may not be either. However with a perfect circle of any size, $\pi = \frac{C}{d}$.

Constructing a New Way to Measure Angles: Now we can finally return back to the concept of measuring angles. We know that for any circle, the relationship between the circumference and radius is exactly the same. So for our new system, a certain angle will be measured by the number of times the radius can fit within the arc borne out by the angle. Observe the following arbitrary angle:



The portion of the circle that fits between the two arrows, is exactly one radius in length around the circle. We will call this measure one **radian**. Now the next natural question to ask is, what number replaces 360? We know that 360 represents the number of degrees in an entire circle, so we naturally ask, how many radians will get us around the entire circle? No matter what size of circle we are dealing with, the number of radians that fit within the circumference, can be found by dividing these values, which leads to: $\frac{C}{r} = 2\pi$. Therefore, the entire circle is 2π radians. In summary: **the radian system measures angles by how many times the radius will fit within the arc of the circle drawn by an angle.**

Converting Between Degrees and Radians: Radians can be hard to visualize, so a powerful tool especially when first dealing with radians to have a way to convert between the different systems. The following two rules help us to do this.

- To convert from degrees to radians, multiply by $\dfrac{\pi}{180}$
- To convert from radians to degrees, multiply by $\dfrac{180}{\pi}$

1. Convert to radians: 90°

2. Convert to radians: 45°

3. Convert to degrees: $\dfrac{\pi}{3}$

4. Convert to degrees: $\dfrac{2\pi}{3}$

Coterminal Angles: We measure angles as an amount of distance covered when traveling counter-clockwise around the circle. So what happens if we go more than one revolution around the circle? We can keep increasing the value of degrees or radians accordingly, but these angles will be the exact same as angles of a smaller value. These angles are called coterminal. For example, the angle $90°$ and the angle $450°$ are the same. This is because $450°$ is $90°$ more than $360°$. We can expand on this concept to create a general method of generating coterminal angles.

Finding Coterminal Angles: Given any angle θ:

- **Degrees:** Add or subtract $360°$ as many times as needed. The result will be coterminal to θ.
- **Radians:** Add or subtract 2π as many times as needed. The result will be coterminal to θ.

5. For the following angles, find two coterminal angles, one which is positive and one which is negative.

(a) $\dfrac{\pi}{3}$

(b) $\dfrac{7\pi}{2}$

(c) 3π

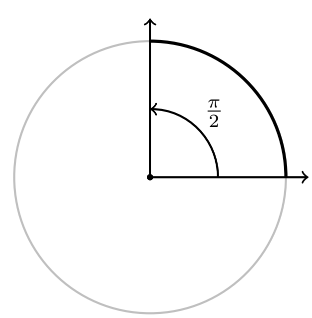
(d) $-\dfrac{\pi}{4}$

Graphing Angles: Graphing the exact position of a radian angle can be difficult, but we can rely on a clever idea. Since the whole circle is 2π radians, we can take any angle θ, and divide $\frac{\theta}{2\pi}$. This will generate a fraction, which represents the proportion of the circle in which the angle θ has traveled. From here, we can estimate the location of θ with decent accuracy (Note: you should ensure that $0 \leq \theta \leq 2\pi$ before using this method. If not, find an angle which is coterminal to θ and meets this condition, then employ this method).

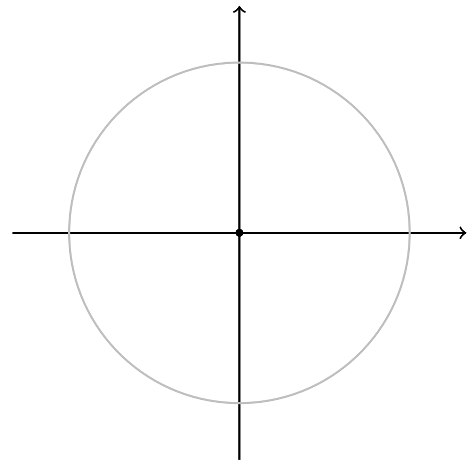
For example, to graph $\frac{\pi}{2}$, we can divide:

$$\frac{\pi/2}{2\pi}$$

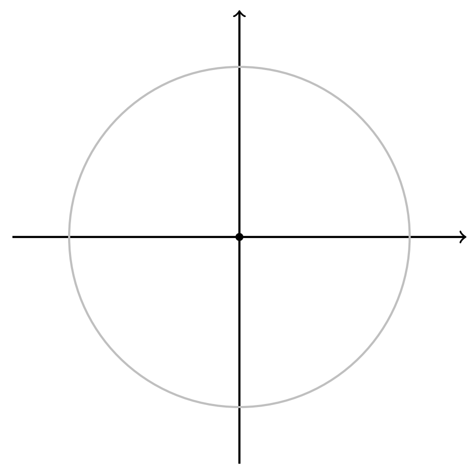
which simplifies to $\frac{1}{4}$. Then, we can graph the angle which travels $\frac{1}{4}$ around the circle. From this we can see that $\frac{\pi}{2}$ is the radian measure analogous to $90°$.



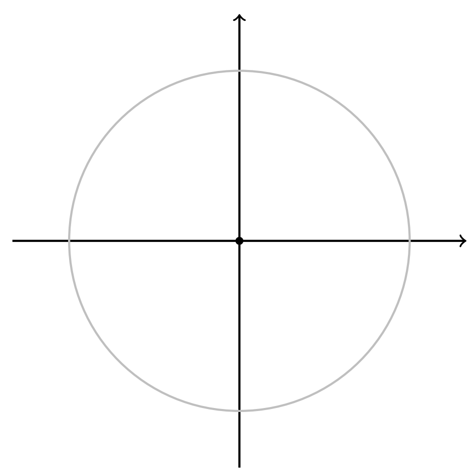
6. Graph the angle: $\frac{5\pi}{6}$



7. Graph the angle: $-\frac{5\pi}{3}$



8. Graph the angle: $\frac{15\pi}{4}$



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