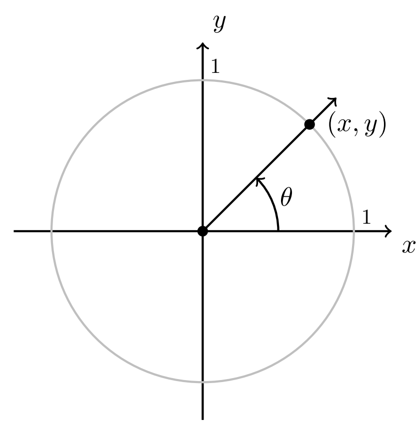
7.2 | The Circular Functions: Sine and Cosine

The Unit Circle: Recall from section 7.1 how every circle is proportional due to the relationship $\frac{C}{d} = \pi$. For this reason, we almost always do work within the *unit circle*. The unit circle is a circle of radius 1 centered at the origin of the xy-plane.

Constructing the Sine and Cosine Functions: With the unit circle in mind, observe that any angle we graph will then have an intersection point with the unit circle which lies in the xy-plane.

This, however, raises the question, how can we find out the (x, y) coordinate of this intersection point? We know how far this point has traveled around the circle in radians, but it would be useful to be able to understand what the corresponding x and y values are for any given angle.

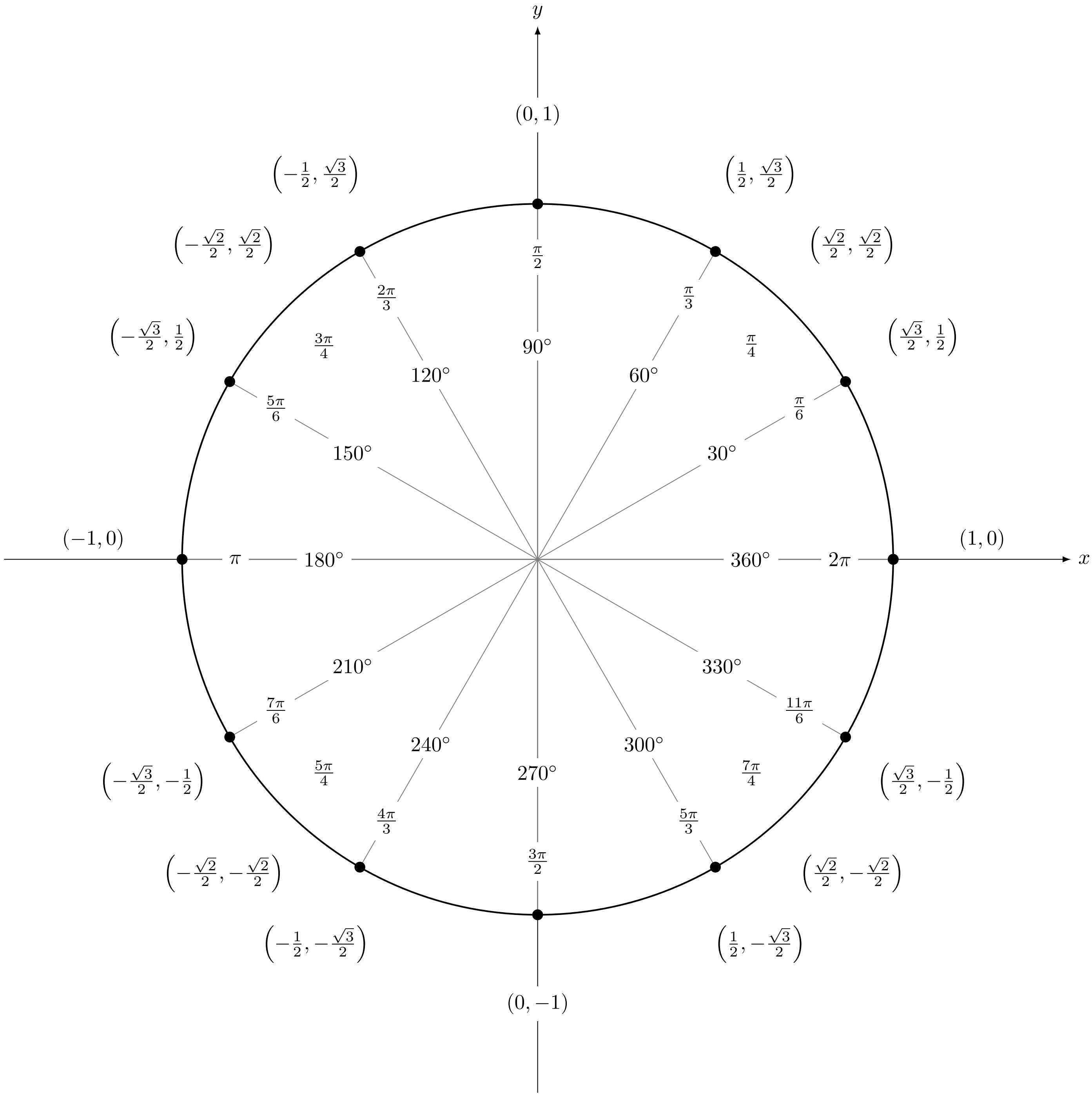


For this reason, we have two new functions, the **sine** function and **cosine** function. Both functions take an angle measure as an input value, but the cosine function outputs the corresponding x value of the (x, y) intersection point, and the sine function outputs the corresponding y value of the (x, y) intersection point. We write $\sin(\theta)$ for the sine function and $\cos(\theta)$ for the cosine function.

Computing Sine and Cosine: Actually finding the value of the sine and cosine functions is more difficult. This is due to the fact that the values that lie on the perimeter of the unit circle are not very nice values. For this reason, humans need to rely on memorization. Common values of sine and cosine as well as their corresponding xy values are included in the unit circle diagram on the following page.

Problem Solving Tip 1. This class will require you to memorize the unit circle. It is a good idea to try and be able to draw it from memory so that when you start a test you can flip the test to the back, draw an entire unit circle, and reference it when needed.

The Unit Circle



1. Find $\sin(\theta)$ and $\cos(\theta)$ if $\theta = \frac{\pi}{4}$

2. Find $\sin(\theta)$ and $\cos(\theta)$ if $\theta = \frac{\pi}{2}$

3. Find $\sin(\theta)$ and $\cos(\theta)$ if $\theta = \frac{2\pi}{3}$

4. Find $\sin(\theta)$ and $\cos(\theta)$ if $\theta = \frac{7\pi}{6}$

5. Find $\sin(\theta)$ and $\cos(\theta)$ if $\theta = \frac{23\pi}{6}$

6. Find $\sin(\theta)$ and $\cos(\theta)$ if $\theta = -\frac{3\pi}{4}$

Problem Solving Tip 2. When asked to find all angles which satisfy a certain equation, remember that for any angle which satisfies an equation, all angles coterminal to it also satisfy the equation. For an angle θ, all angles coterminal to θ are of the form $\theta + 2\pi k$, where k is any integer. Use this form to describe your solutions.

7. Find all angles which satisfy the given equation: $\sin(\theta) = \frac{1}{2}$

8. Find all angles which satisfy the given equation: $\cos(\theta) = -\frac{\sqrt{3}}{2}$

9. Find all angles which satisfy the given equation: $\sin(\theta) = 0$

10. Find all angles which satisfy the given equation: $\cos(\theta) = -1$

11. Find all angles which satisfy the given equation: $\cos(\theta) = \frac{\sqrt{2}}{2}$

Terminal Side: The terminal side of an angle is the side of the angle which faces the x-axis. This can be confusing to understand. Imagine opening a laptop slightly so that it makes a 45° angle with the keyboard. The terminal side of the angle would be the side that has the screen of the laptop as it is closest to the keyboard (x-axis). Alternatively, assume you have a flexible enough laptop to open it at a wide 135° angle. The terminal side would be the back of the laptop screen as it is closer to the table (x-axis).

Standard Position: Standard position simply implies that an angle is graphed at the origin on the xy-plane.

Reference Angle: For a given angle θ, the reference angle (usually named α) is the acute angle formed between the terminal side of θ and the x-axis.

Quadrants: We split the xy-plane into 4 quadrants. The quadrants are named using roman numerals as follows:

- **I**: Positive x, positive y
- **II**: Negative x, positive y
- **III**: Negative x, negative y
- **IV**: Positive x, negative y

Textbook Theorem 7.1. Reference Angle Theorem. Suppose α is the reference angle for θ. Then:

$$\cos(\theta) = \pm\cos(\alpha) \text{ and } \sin(\theta) = \pm\sin(\alpha)$$

where the choice of the ($\pm$) depends on the quadrant in which the terminal side of θ lies.

Textbook Theorem 7.2. Two angles α and β are coterminal if and only if:

$$\cos(\alpha) = \cos(\beta) \text{ and } \sin(\alpha) = \sin(\beta)$$

The unit circle can be used to find points in the xy-plane that generate a desired angle, but what happens if we wish to go beyond the unit circle and find angles with other coordinate pairs? The following theorem helps us to calculate the angles of points not on the unit circle.

Textbook Theorem 7.3. If $Q(x, y)$ is the point on the terminal side of an angle θ, plotted in standard position, which lies on the circle $x^2 + y^2 = r^2$ then $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Moreover,

$$\cos(\theta) = \frac{x}{r} = \frac{x}{\sqrt{x^2+y^2}} \quad \text{and} \quad \sin(\theta) = \frac{y}{r} = \frac{y}{\sqrt{x^2+y^2}}$$

In English 7.3. Given a point $Q(x, y)$ on the angle θ, the following relationships can be used to find values of $\sin(\theta)$ and $\cos(\theta)$:

- $x^2 + y^2 = r^2$
- $r = \sqrt{x^2 + y^2}$
- $\cos(\theta) = \dfrac{x}{r}$
- $\sin(\theta) = \dfrac{y}{r}$

Use the first two equations to find all three values x, y, and r. Then, use these values to compute $\sin(\theta)$ and $\cos(\theta)$ accordingly.

12. **Worked Example:** Let θ be the angle containing the point $Q(3, 4)$. Find $\sin(\theta)$ and $\cos(\theta)$.

Scan the QR code for a video solution

13. Let θ be the angle containing the point $P(-7, 24)$. Find $\sin(\theta)$ and $\cos(\theta)$.

14. **Worked Example:** If $\sin(\theta) = -\dfrac{7}{25}$ with θ in Quadrant IV, what is $\cos(\theta)$?

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15. If $\cos(\theta) = \dfrac{4}{9}$ with θ in Quadrant I, what is $\sin(\theta)$?

16. If $\sin(\theta) = \dfrac{5}{13}$ with θ in Quadrant II, what is $\cos(\theta)$?