8.2 | More Identities

In this section, we focus on identities that can be built from operating on the angle θ within a trig function. This starts with the following theorem:

Textbook Theorem 8.4. Even / Odd Identities: For all applicable angles θ,

- $\cos(-\theta) = \cos(\theta)$
- $\sin(-\theta) = -\sin(\theta)$
- $\tan(-\theta) = -\tan(\theta)$
- $\sec(-\theta) = \sec(\theta)$
- $\csc(-\theta) = -\csc(\theta)$
- $\cot(-\theta) = -\cot(\theta)$

1. Verify the identity: $\sin(3\pi - 2\theta) = -\sin(2\theta - 3\pi)$

2. Verify the identity: $\cos\left(-\frac{\pi}{4} - 5t\right)$

Looking at the graphs of sine and cosine, you may have realized that if you set the phase shift of sine to a certain value, it will perfectly overlap with that of cosine. The following theorem focuses on this idea; it provides a way to convert one trig function into another by using a phase shift.

Textbook Theorem 8.6. Cofunction Identities: For all applicable identities θ,

- $\cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$
- $\sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta)$
- $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$
- $\sin\left(\frac{\pi}{2} - \theta\right) = \cos(\theta)$
- $\csc\left(\frac{\pi}{2} - \theta\right) = \sec(\theta)$
- $\cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta)$

The book also introduces theorems 8.5 and 8.7, however both are combined into one cohesive theorem as follows:

Textbook Theorem 8.8. Sum and Difference Identities: For all applicable angles α and β,

- $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$
- $\sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$
- $\tan(\alpha \pm \beta) = \dfrac{\tan(\alpha) \pm \tan(\beta)}{1 \mp \tan(\alpha)\tan(\beta)}$

Note About Theorem 8.8: When reading the $\pm$ or $\mp$ signs, simply pick either the top or bottom sign and read the corresponding top or bottom sign for the entire identity. For example, if you want to use the cosine addition identity, read the top signs, if you want to use the cosine minus identity, read the bottom signs.

Problem Solving Tip 1. Finding Values of θ not on the Unit Circle
If asked to find a value of a trig function for an angle θ which does not appear on the unit circle, one method is to break up the angle theta into a sum: $\alpha \pm \beta = \theta$, where α and β are angles present on the unit circle. Then you can use the sum and difference identities to expand the function into something that is solvable.

3. **Worked Example:** Find the value of $\cos\left(\frac{13\pi}{12}\right)$

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4. Find the value of $\sin\left(\frac{11\pi}{12}\right)$

5. Find the value of $\tan\left(\frac{17\pi}{12}\right)$

6. If α is a Quadrant IV angle with $\cos(\alpha) = \frac{\sqrt{5}}{5}$, and $\sin(\beta) = \frac{\sqrt{10}}{10}$, where $\frac{\pi}{2} < \beta < \pi$, find: $\cos(\alpha + \beta)$.
Hint: Use the equations $x^2 + y^2 = r^2$, $\cos(\theta) = \frac{x}{r}$, and $\sin(\theta) = \frac{y}{r}$.

In addition to the sum and difference identities, we also have identities which involve multiplication and division, as well as one which helps reduce exponentials.

Textbook Theorem 8.9. Double Angle Identities: For all applicable angles θ,

- $\cos(2\theta) = \begin{cases} \cos^2(\theta) - \sin^2(\theta) \\ 2\cos^2(\theta) - 1 \\ 1 - 2\sin^2(\theta) \end{cases}$
- $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- $\tan(2\theta) = \dfrac{2\tan(\theta)}{1 - \tan^2(\theta)}$

Textbook Theorem 8.10. Power Reduction Formulas: For all angles θ,

- $\cos^2(\theta) = \dfrac{1 + \cos(2\theta)}{2}$
- $\sin^2(\theta) = \dfrac{1 - \cos(2\theta)}{2}$

Textbook Theorem 8.11. Half Angle Formulas: For all applicable angles θ,

- $\cos\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1 + \cos(\theta)}{2}}$
- $\sin\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1 - \cos(\theta)}{2}}$
- $\tan\left(\dfrac{\theta}{2}\right) = \pm\sqrt{\dfrac{1 - \cos(\theta)}{1 + \cos(\theta)}}$

where the choice of $\pm$ depends on the quadrant in which the terminal side of $\dfrac{\theta}{2}$ lies.

7. **Worked Example:** Use the Half Angle Formulas to find the value of $\sin\left(\frac{\pi}{12}\right)$

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8. Use the Half Angle Formulas to find the value of $\cos\left(\frac{\pi}{8}\right)$

9. If $\sin(\theta) = -\frac{7}{25}$ where $\frac{3\pi}{2} < \theta < 2\pi$, find the exact value of $\cos(2\theta)$.

10. If $\cos(\theta) = \frac{12}{13}$ where $\frac{3\pi}{2} < \theta < 2\pi$, find the exact value of $\sin\left(\frac{\theta}{2}\right)$.

For the following identities, assume all quantities are defined.

11. **Worked Example:** Verify: $(\cos(\theta) + \sin(\theta))^2 = 1 + \sin(2\theta)$

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12. Verify: $8\sin^4(\theta) = \cos(4x) - 4\cos(2x) + 3$
    Hint: Use the fact that $4x = 2(2x)$ *and* $x^4 = (x^2)^2$

13. Verify: $\dfrac{1}{\cos(\theta) - \sin(\theta)} - \dfrac{1}{\cos(\theta) + \sin(\theta)} = \dfrac{2\sin(\theta)}{\cos(2\theta)}$